ENGINEERING MATHEMATICS

BY

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PREFACE

This book is intended for engineering students who have completed the study of the elementary calculus and for graduate engineers seeking to bolster their present knowledge of mathematics. Its purpose is two-fold. It is intended to strengthen the student in algebra and to provide him with certain mathematical tools which depend on the calculus.

The topics covered in this book with the exception of the chapters on vector algebra made up the subject matter of a course taught by the author for some years to engineering students at Cornell University. The sections on vector algebra were added as a result of suggestions made by college graduates who were using the author's notes in conjunction with a calculus text to review and strengthen their college mathematics. College graduates working in the fields of engineering, statistics, physics and meteorology will find much of interest in this book. Chapters 13 and 14 are special applications of the material of Chapters 7 and 10 and are primarily for physicists and electrical engineers.

The theory of determinants comes early in the book, being taken up in the second chapter. Determinants and matrix theory are used throughout the remainder of the book wherever it seems advisable to do so.

The treatment of the algebra is unusual in places since it is possible to use the concept of limit and derivative to simplify some of the proofs. The method of identifying multiple roots of algebraic equations has been systematized by the author more completely than is done in most algebra books. Tests for integer roots and rational roots have been omitted with more emphasis being placed on methods of approximating all the roots whether rational or irrational.

The analysis of the result of an approximate Fourier analysis is an original work by the author. The conclusion that an approximate analysis based on the trapezoidal rule is more to be relied on than one based on Simpson's rule will be a surprise to many readers.

Certain sections containing elementary material have been marked with an asterisk. This has been done for the convenience of engineering graduates using the text for review. Such students might safely skip most of the marked sections.

HARRY SOHON

Philadelphia, Pa.
May, 1944

CONTENTS

CHAPTER 1

INTERPOLATION FORMULAS

1.1 Interpolation. Interpolation is the process of finding approximately, from given values of a function, other intermediate values. One usually represents the function by a polynomial whose curve passes through some of the given points. The polynomial curve may be a straight line passing through two points, a parabola passing through three points, a third degree curve passing through four points, etc. If the given values are for arguments close together in comparison with the change in the value of the function, as in the usual table of common logarithms, and tables of trigonometric functions, the straight line between given points is quite satisfactory. This is interpolation by proportional parts. If the given values are for arguments far apart, as in tables of hyperbolic functions, Bessel's functions, etc., it is essential that the polynomial be at least of the second degree.

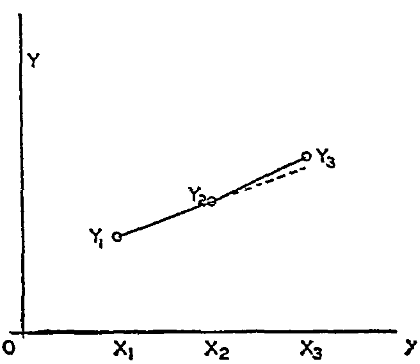

Fig. 1–1

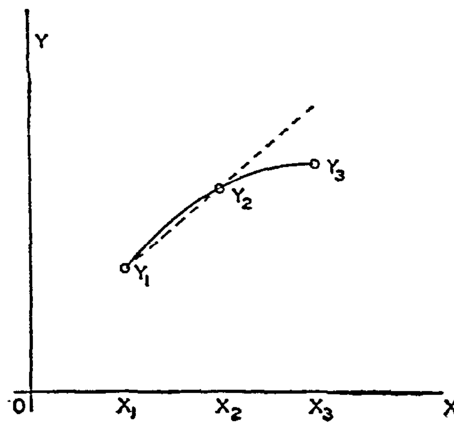

Fig. 1–2

1.2 Figure 1–1 shows three consecutive points plotted from a table that requires interpolation only by proportional parts. Note that the straight line through two points passes very near to the third point. Figure 1–2 shows three consecutive points plotted from a table in which interpolation by proportional parts is not satisfactory. It would be much better to pass a smooth curve through the three points in Fig. 1–2. On the other hand the increased accuracy obtained by passing a smooth curve through the three points in Fig. 1–1 would not be worth the trouble.

1.3 The comparison in the preceding paragraph is illustrated further by the following numerical example. A table of common logarithms gives: $\log 1.10 = 0.0414$, $\log 1.11 = 0.0453$, and $\log 1.12 = 0.0492$. Now 1.11 is halfway between 1.10 and 1.12 and 0.0453 is halfway between 0.0414 and 0.0492; therefore, a straight line relation is satisfactory. Another table gives: $\log 1.100 = 0.041393$, $\log 1.110 = 0.045323$, and $\log 1.120 = 0.049218$. Now 1.110 is halfway between 1.100 and 1.120, but 0.045305 instead of 0.045323 is halfway between 0.041393 and 0.049218; therefore, in this case the straight line assumption is not justified. If a second degree curve is assumed to pass through the points 1.100, 1.120, and 1.140 ($\log 1.140 = 0.056905$), this curve gives $\log 1.110 = 0.045323$ which checks to the last figure.

1.4 The equation of the polynomial which we pass through a certain number of the given points is the **interpolation formula.** The equation of the polynomial can be written several ways, but it is still the same polynomial. However, if we do not use all the terms in the polynomial, then we find that the arrangement of the terms makes a difference. This accounts for the various interpolation formulas known as Newton's formula, Stirling's formula, Bessel's formula, and LaGrange's formula. We find a third, fourth, or fifth degree polynomial which approximates the function, then we find a first, second, or third degree polynomial which approximates the approximation. This sounds as if we are making things difficult, but if we require a value between two given values where the increments are large this is the easiest way to find it.

1.5 Differences. Let y be a function of x, several values of which are tabulated. Suppose successive values of x differ by the constant value ω so that $x_1 = x_0 + \omega$, $x_2 = x_1 + \omega$, etc. Suppose y_0 corresponds to x_0, y_1 to x_1, etc. Let $y_1 - y_0 = a_1$, $y_2 - y_1 = a_2$, etc. We can tabulate x, y, and a as follows:

TABLE I–1

x	y	a
x_0	y_0	
		a_1
x_1	y_1	
		a_2
x_2	y_2	
		a_3
x_3	y_3	

Each value of a is inserted in the line between the two values of y whose difference gives that value of a. Let $b_1 = a_2 - a_1$, $b_2 = a_3 - a_2$, etc. Then b gives the differences of the values of a which in turn are the differ-

ences of the tabulated values of the function. The a's are called **first differences**, the b's are called **second differences**, and differences of the second differences are called **third differences**, etc. The symbol for first differences is Δ^1, the symbol for second differences is Δ^2, and Δ^n is the symbol for nth order differences. In Table I–2 we have extended the entries, and included higher order differences. Each difference is obtained by subtracting the number above to the left from the number below to the left.

TABLE I–2

x	y	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-4}	y_{-4}						
		a_{-4}					
x_{-3}	y_{-3}		b_{-3}				
		a_{-3}		c_{-3}			
x_{-2}	y_{-2}		b_{-2}		d_{-2}		
		a_{-2}		c_{-2}		e_{-2}	
x_{-1}	y_{-1}		b_{-1}		d_{-1}		f_{-1}
		a_{-1}		c_{-1}		e_{-1}	
x_0	y_0		b_0		d_0		f_0
		a_1		c_1		e_1	
x_1	y_1		b_1		d_1		f_1
		a_2		c_2		e_2	
x_2	y_2		b_2		d_2		
		a_3		c_3			
x_3	y_3		b_3				
		a_4					
x_4	y_4						

1.6 The interpolation formula is a systematic method of representing a function by a polynomial. To make such a representation efficiently we should first decide what degree polynomial is required. The manner in which the use of difference tables guide us is illustrated by consideration of the tables corresponding to some simple expressions.

TABLE I–3

x	$y = x$	Δ^1	Δ^2
0	0		
		1	
1	1		0
		1	
2	2		0
		1	
3	3		0
		1	
4	4		0
		1	
5	5		

TABLE I–4

x	$y = x^2$	Δ^1	Δ^2	Δ^3
0	0			
		1		
1	1		2	
		3		0
2	4		2	
		5		0
3	9		2	
		7		0
4	16		2	
		9		
5	25			

TABLE I–5

x	$y = x^3$	Δ^1	Δ^2	Δ^3	Δ^4
0	0				
		1			
1	1		6		
		7		6	
2	8		12		0
		19		6	
3	27		18		0
		37		6	
4	64		24		
		61			
5	125				

In Table I–3, y is of the first degree in x and we see that the first differences are equal and higher differences zero. In Table I–4, y is of the second degree in x and the second differences are equal. Table I–5 shows the case of $y = x^3$ where the third differences are equal. In general the student will find if y is a polynomial of the nth degree in x that the nth order differences are all equal and higher order differences are zero. The proof of this statement is left to the student.

If there is no column of differences all equal, then y cannot be expressed as a polynomial in x. An interesting example of such a function is $y = 2^x$. The difference table is shown below.

TABLE I–6

x	$y = 2^x$	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
0	1					
		1				
1	2		1			
		2		1		
2	4		2		1	
		4		2		1
3	8		4		2	
		8		4		
4	16		8			
		16				
5	32					

Table I–7 is the type usually encountered. An important question in making up a difference table is how far to carry it. If all the items in one column are equal, all the higher differences will be zero. In Table I–7, the third differences are quite scattered and seem to be unreliable. Let us see whether they really should be discarded. The value of y given for $x =$ 1.08 is 0.9597; by this we understand that y is between 0.95965 and 0.95975;

any value of y in the table is understood to have an uncertainty of one half in the last place. If one value of y happens to be too big and the following value too small, the difference might be off by unity in the last place. Similarly the second differences have an uncertainty of two in the last place and the third differences of four in the last place. The second differences are 0.0008 ± 0.0002, etc., while the third differences are -0.0001 ± 0.0004, etc. Therefore, in the example of Table I–7, all third differences listed may be considered zero.

TABLE I–7

x	y	Δ^1	Δ^2	Δ^3
1.00	1.0000			
		−0.0112		
1.02	0.9888		0.0008	
		−0.0104		−0.0001
1.04	0.9784		0.0007	
		−0.0097		0.0000
1.06	0.9687		0.0007	
		−0.0090		0.0000
1.08	0.9597		0.0007	
		−0.0083		−0.0002
1.10	0.9514		0.0005	
		−0.0078		0.0001
1.12	0.9436		0.0006	
		−0.0072		0.0000
1.14	0.9364		0.0006	
		−0.0066		−0.0001
1.16	0.9298		0.0005	
		−0.0061		
1.18	0.9237			

1.7 Newton's Formula. In order to get a formula for y in terms of x such that $y = y_0$ when $x = x_0$, $y = y_1$ when $x = x_1$, etc., we can write

$$y = A_0 + (x - x_0)A_1 + (x - x_0)(x - x_1)A_2 + (x - x_0)(x - x_1)(x - x_2)A_3 \cdots.$$

When $x = x_0$ all the terms but the first vanish and we have $y_0 = A_0$. If $x = x_1$ every term but the first and second vanish and we now have

$$y_1 = y_0 + (x_1 - x_0)A_1.$$

Now $y_1 - y_0 = a_1$ and $x_1 - x_0 = \omega$; therefore,

$$A_1 = \frac{a_1}{\omega}\cdot$$

If $x = x_2$, every term but the first three vanish and we have

$$y_2 = y_0 + (x_2 - x_0)\frac{a_1}{\omega} + (x_2 - x_0)(x_2 - x_1)A_2.$$

But

$$y_2 - y_0 = (y_2 - y_1) + (y_1 - y_0) = a_2 + a_1$$

and

$$x_2 - x_0 = 2\omega, \quad \text{and} \quad x_2 - x_1 = \omega$$

therefore,

$$a_2 + a_1 = 2a_1 + 2\omega^2 A_2.$$

This can be written

$$2\omega^2 A_2 = a_2 - a_1 = b_1.$$

Therefore,

$$A_2 = \frac{b_1}{2\omega^2}\cdot$$

When we let $x = x_3$ we have

$$y_3 = y_0 + (x_3 - x_0)\frac{a_1}{\omega} + (x_3 - x_0)(x_3 - x_1)\frac{b_1}{2\omega^2}$$

$$+ (x_3 - x_0)(x_3 - x_1)(x_3 - x_2)A_3.$$

When this is solved for A_3 we find

$$A_3 = \frac{c_2}{3!\omega^3}\cdot$$

By the same process we find

$$A_4 = \frac{d_2}{4!\omega^4}\cdot$$

If k is the fractional distance between entries desired, then

$$\frac{x - x_0}{\omega} = k, \quad \frac{x - x_1}{\omega} = k - 1, \quad \frac{x - x_2}{\omega} = k - 2 \cdots$$

We can now write Newton's formula

$$y = y_0 + ka_1 + \frac{k(k-1)}{2!}b_1 + \frac{k(k-1)(k-2)}{3!}c_2$$

$$+ \frac{k(k-1)(k-2)(k-3)}{4!}d_2 + \cdots.$$

The terms taken from the difference table are in the diagonal a_1, b_1, c_2, d_2, e_3, f_3, etc. This formula is intended to enable us to obtain values for the function between $x = x_0$ and $x = x_1$ with the help of the values at $x = x_2$ and $x = x_3$, etc.

1.8 As an example of the use of Newton's formula let us find y corresponding to $x = 1.01$ in Table I–7. We read from the table $y_0 = 1.0000$, $a_1 = -0.0112$, $b_1 = 0.0008$, higher order differences being negligible. $x = 1.01$ is halfway between 1.00 and 1.02; therefore, $k = 0.5$; we now have

$$y = 1.0000 - (0.5)(0.0112) + \frac{(0.5)(-0.5)}{2}(0.0008)$$

$$= 1.0000 - 0.0056 - 0.0001 = 0.9943.$$

1.9 In the preceding paragraph the value of y corresponding to $x = 1.01$ in Table I–7 was found using first and second difference terms in Newton's formula. In order to obtain the difference terms for use in the formula, we had to make use of the values of y corresponding to the following four values of x: 1.00, 1.02, 1.04, and 1.06. Suppose we now require the value of y corresponding to $x = 1.062$. Newton's formula could be used again and we would obtain the difference terms from the values of y corresponding to the following values of x: 1.06, 1.08, 1.10, and 1.12. However, since we require a value corresponding to a value of x very close to 1.06, it is reasonable to suggest that $x = 1.04$ is as important as $x = 1.08$, and $x = 1.02$ is as important as $x = 1.10$ in finding the approximate value of y.

Referring to Table I–2, to obtain a value close to x_0, a_{-1} is as important as a_1 (we use their average and call it a_0). Instead of using b_1 we should use b_0, the average value of c_{-1} and c_1 (called c_0) instead of c_2, and d_0 instead of d_2, etc. When Newton's formula is rewritten to contain a_0, b_0, c_0, etc. instead of a_1, b_1, c_2, etc., it is known as Stirling's formula.

1.10 Stirling's Formula. We shall now derive Stirling's formula from Newton's formula by substituting for a_1, b_1, c_2, etc., their values in terms of a_0, b_0, c_0, etc., where $a_0 = 0.5\ (a_{-1} + a_1)$, $c_0 = 0.5(c_{-1} + c_1)$, etc., and b_0, d_0, etc. are as defined in the difference table, Table I–2.

A simple way of deriving the first four terms of Stirling's formula is as follows. Since only four terms are required the highest order difference in the result will be third differences. Then assume that fourth and higher order differences in the table are zero and that third order differences are all equal. This assumption is only for the purpose of deriving four terms

of Stirling's formula. We must find from the table a_1, b_1, and c_2 in terms of a_0, b_0, and c_0.

$$a_1 = a_0 + \frac{b_0}{2}$$

$$b_1 = b_0 + c_1 = b_0 + c_0$$

$$c_2 = c_0$$

Substitution in the first four terms of Newton's formula gives

$$y = y_0 + ka_1 + \frac{k(k-1)}{2!} b_1 + \frac{k(k-1)(k-2)}{3!} c_2$$

$$= y_0 + ka_0 + k\frac{b_0}{2} + \frac{k^2 - k}{2!} b_0 + \frac{k^2 - k}{2!} c_0 + \frac{k^3 - 3k^2 + 2k}{3!} c_0.$$

$$y = y_0 + ka_0 + \frac{k^2}{2!} b_0 + \frac{k(k^2 - 1)}{3!} c_0.$$

This gives the first four terms of Stirling's formula which is used to get values for y corresponding to values of x close to x_0.

1.11 We cannot use Stirling's formula to get the value of y corresponding to $x = 1.01$ in Table I–7 since we are unable to compute a_0, b_0, etc., at the end of a table. Newton's formula is used at the two ends of a table. To illustrate the use of Stirling's formula let us find the value of y corresponding to $x = 1.062$ in Table I–7. We find from the table

$$y_0 = 0.9687$$

$$a_0 = 0.5(-0.0097 - 0.0090) = -0.00935$$

$$b_0 = 0.0007$$

$$k = \frac{0.002}{0.020} = 0.1$$

$$y = 0.9687 + (0.1)(-0.00935) + \frac{(0.1)^2}{2}(0.0007)$$

$$= 0.9687 - 0.0009 = 0.9678$$

1.12 Bessel's Formula. Bessel's interpolation formula is used to find the value of y corresponding to a value of x near the middle of the interval between $x = x_0$ and $x = x_1$. It is written in terms of a_1, c_1, e_1, etc., as given in Table I–2, and in terms of b (the average of b_0 and b_1), d (the average of d_0 and d_1), etc. This means that x_0 and x_1 are depended on to the same extent, x_{-1} and x_2 are depended on together, and x_{-2} and x_3 have the same

weight in determining the required value of y. It is evident then that Bessel's formula can be used neither at the beginning of a table nor at the end of a table. Newton's formula would be used at the ends of a table.

Bessel's formula can be derived from Newton's formula by a process similar to that used to derive Stirling's formula. We require a_1, b_1, and c_2 in terms of a_1, b, and c_1 to obtain the first four terms of Bessel's formula. The result is

$$y = y_0 + ka_1 + \frac{k(k-1)}{2!} b + \frac{k(k-1)(k-0.5)}{3!} c_1.$$

Note that in those cases in which we interpolate using proportional parts, we are really using the first two terms of Newton's formula or Bessel's formula.

1.13 Derivatives. The derivatives of a tabulated function can be found very readily with interpolation formulas. Usually it will be found more satisfactory to use an interpolation formula than to plot the function and measure the slope of the curve graphically. When finding the derivative of a function that has been obtained experimentally the value chosen for ω must be made as large as possible to minimize the effect of errors of observation.

The formula for the derivative is obtained by differentiating one of the interpolation formulas. Since $x = x_0 + k\omega$, we have

$$\frac{dy}{dx} = \frac{dy}{dk}\frac{dk}{dx} = \frac{1}{\omega}\frac{dy}{dk}.$$

We need only differentiate the formula with respect to k and divide by ω. Newton's formula gives

$$y = y_0 + ka_1 + \frac{k^2 - k}{2} b_1 + \frac{k^3 - 3k^2 + 2k}{6} c_2 + \cdots$$

$$\frac{dy}{dx} = \frac{1}{\omega}\left[a_1 + \frac{2k-1}{2} b_1 + \frac{3k^2 - 6k + 2}{6} c_2 + \cdots\right]$$

at $x = x_0$, $k = 0$, and we have

$$\left(\frac{dy}{dx}\right)_0 = \frac{1}{\omega}\left[a_1 - \frac{b_1}{2} + \frac{c_2}{3} \cdots\right]$$

at $x = x_0 + \dfrac{\omega}{2}$, $k = 0.5$, and we have

$$\left(\frac{dy}{dx}\right)_{0.5} = \frac{1}{\omega}\left[a_1 - \frac{c_2}{24} \cdots\right].$$

To illustrate the use of these formulas let us find the derivative at $x = 1.00$ for the function tabulated in Table I-7. The table gives the following values: $a_1 = -0.0112$, $b_1 = 0.0008$, c_2 is negligible, and $\omega = 0.02$;

$$\frac{dy}{dx} = \frac{1}{0.02}\left[-0.0112 - \frac{0.0008}{2}\right] = (50)(-0.0116) = -0.5800.$$

1.14 If Stirling's formula is differentiated with respect to k and divided by ω we have

$$\frac{dy}{dx} = \frac{1}{\omega}\left[a_0 + kb_0 + \frac{3k^2 - 1}{6} c_0\right]$$

at $x = x_0$, $k = 0$, and Stirling's formula gives

$$\left(\frac{dy}{dx}\right)_0 = \frac{1}{\omega}\left[a_0 - \frac{c_0}{6}\right].$$

Stirling's formula is not used near $k = 0.5$; therefore, we shall not substitute $k = 0.5$ in the expression above.

1.15 The first three terms of Bessel's formula for the derivative is found by the method outlined for Newton's formula to be

$$\frac{dy}{dx} = \frac{1}{\omega}\left[a_1 + \frac{2k - 1}{2} b + \frac{6k^2 - 6k + 1}{12} c_1\right].$$

Bessel's formula is not used near $k = 0$, but at $k = 0.5$ we have

$$\left(\frac{dy}{dx}\right)_{0.5} = \frac{1}{\omega}\left[a_1 - \frac{c_1}{24}\right].$$

1.16 Higher Derivatives. Higher derivatives are found by the same method that was used for first derivatives. For the second derivative differentiate twice with respect to k and divide by ω^2. For the third derivative differentiate three times with respect to k and divide by ω^3.

1.17 LaGrange's Formula. The three interpolation formulas already discussed can be used only when values of the argument x increase by equal steps. If neither x nor y increase by equal steps we can use LaGrange's formula. LaGrange's formula is a systematic form of writing a polynomial passing through any fixed number of given points. Instead of giving the general formula we give the formula for the special case of a curve passing through four points. The extension of the formula to include more than four points is obvious.

$$y = \frac{(x - x_2)(x - x_3)(x - x_4)y_1}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + \frac{(x - x_1)(x - x_3)(x - x_4)y_2}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}$$
$$+ \frac{(x - x_1)(x - x_2)(x - x_4)y_3}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{(x - x_1)(x - x_2)(x - x_3)y_4}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}.$$

This can be differentiated to obtain derivatives. The expressions for the derivatives do not simplify in general. The first derivative at $x = x_1$ is

$$\left(\frac{dy}{dx}\right)_1 = \frac{y_1}{x_1 - x_2} + \frac{y_1}{x_1 - x_3} + \frac{y_1}{x_1 - x_4} + \frac{(x_1 - x_3)(x_1 - x_4)y_2}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)}$$
$$+ \frac{(x_1 - x_2)(x_1 - x_4)y_3}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{(x_1 - x_2)(x_1 - x_3)y_4}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}.$$

1.18 Taylor's Theorem. In some cases Taylor's theorem provides us with an expression that can be used easily as an interpolation formula.

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots.$$

If we are interpolating in a table of sines and cosines the first two terms of Taylor's series become

$$\sin x = \sin x_0 + (x - x_0) \cos x_0$$
$$\cos x = \cos x_0 - (x - x_0) \sin x_0$$

where x and x_0 are in radians. Since $\sin x_0$ and $\cos x_0$ will usually be found tabulated side by side, interpolation this way is quite simple.

1.19 Inverse Interpolation. If it is necessary to take second or higher order differences into account when finding the value of y corresponding to a given value of x, then it is of course necessary to take higher order differences into account when finding the value of x corresponding to a given value of y. The various interpolation formulas discussed above do not lead to the inverse process very readily. The simplest way to do inverse interpolation is to make an approximation to the value of x (perhaps using proportional parts), then find y for the approximate value of x using the higher order differences. It may be that the approximated value of x is close enough to a tabulated value so that interpolation by proportional parts (first differences) is satisfactory between them. Otherwise another approximation is necessary. It is possible to use LaGrange's formula for backward interpolation but it is usually better to do as suggested above.

If a lot of interpolation must be done in a table it is worthwhile to keep a record of interpolated values. In this way a new table will be compiled in which tabulated values may be close enough for interpolation by proportional parts.

PROBLEMS ON CHAPTER 1

1. Prove that if y is a polynomial of the nth degree in x and if y is tabulated for equal increments of x that the nth order differences will be equal.
2. Derive the value for A_4 given in section 1.7 for Newton's formula.

3. Derive the fifth term in Stirling's interpolation formula, that is, the term containing the fourth difference d_0.

4. Derive the fifth term in Bessel's formula, that is, the term containing the fourth difference d.

5. In the following table find the values of y corresponding to the following values of x: 0.603; 0.604; 0.610; 0.615.

x	y	z
0.600	0.56464	0.82534
0.620	0.58104	0.81388
0.640	0.59720	0.80210
0.660	0.61312	0.78999
0.680	0.62879	0.77757
0.700	0.64422	0.76484

6. Find z corresponding to the following values of x in the table above: 0.604; 0.605; 0.615.

7. Find y corresponding to the following values of x; in each case use the appropriate interpolation formula: 0.602; 0.615; 0.622; 0.629.

8. Find z corresponding to each value of x in problem 7; in each case use the appropriate formula.

9. Derive the fourth difference term for Stirling's formula for the derivative at $x = x_0$.

10. Derive the fourth difference term for Bessel's formula for the derivative halfway between x_0 and x_1.

11. Derive the first three terms of Newton's formula for the second derivative at x_0.

12. Find dy/dx corresponding to each value of x tabulated in the above table.

13. Find dz/dx corresponding to each value of x tabulated in the above table.

14. What does the solution to problems 12 and 13 indicate when compared with the values tabulated?

15. Write LaGrange's formula for a curve passing through three given points.

16. Write LaGrange's formula for a curve passing through five given points.

17. Write LaGrange's formula for the first derivative of a curve passing through five given points.

18. Find the value of x corresponding to the following values of y: 0.58510; 0.59559.

19. Find the value of x corresponding to the following values of z: 0.82000; 0.80000.

CHAPTER 2

DETERMINANTS

2.1 Introduction. The analysis of many problems in astronomy, mechanics, electrical circuits, and many other phases of engineering leads to sets of simultaneous linear algebraic equations in several unknowns. In order to systematize the solution of such a set of equations the early investigators, notably Leibnitz and Cramer, invented what may be called determinant analysis.

The two equations in x and y

$$a_{11}x + a_{12}y = k_1,$$

$$a_{21}x + a_{22}y = k_2$$

have for solution

$$x = \frac{k_1a_{22} - k_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}},$$

$$y = \frac{a_{11}k_2 - a_{21}k_1}{a_{11}a_{22} - a_{21}a_{12}}$$

provided $a_{11}a_{22} - a_{12}a_{21}$ is not zero. For example, the solution of the equations

$$\begin{aligned} 3x + 4y &= 5, \\ x - 2y &= 8 \end{aligned}$$

can be found by substituting in the expressions above

$$x = \frac{-10 - 32}{-6 - 4} = 4.2,$$

$$y = \frac{24 - 5}{-6 - 4} = -1.9.$$

The three equations in x, y, and z

$$a_{11}x + a_{12}y + a_{13}z = k_1,$$
$$a_{21}x + a_{22}y + a_{23}z = k_2,$$
$$a_{31}x + a_{32}y + a_{33}z = k_3,$$

have for solution

$$x = \frac{k_1a_{22}a_{33} + k_2a_{32}a_{13} + k_3a_{12}a_{23} - k_1a_{32}a_{23} - k_2a_{12}a_{33} - k_3a_{22}a_{13}}{a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}},$$

$$y = \frac{1}{D}[a_{11}k_2a_{33} + a_{21}k_3a_{13} + a_{31}k_1a_{23} - a_{11}k_3a_{23} - a_{21}k_1a_{33} - a_{31}k_2a_{13}],$$

$$z = \frac{1}{D}[a_{11}a_{22}k_3 + a_{21}a_{32}k_1 + a_{31}a_{12}k_2 - a_{11}a_{32}k_2 - a_{21}a_{12}k_3 - a_{31}a_{22}k_1],$$

where D is the denominator in the expression above for x and $D \neq 0$.

If we had eight equations in eight unknowns the solution would be written in fraction form where the numerator and denominator each contained over 40,000 terms! Such a formula of course would be useless unless we could be sure that most of the terms were negligible, or unless we had some simplifying process. The purpose of this chapter is to explain the simplifying methods, and to tell what to do when the expression that belongs in the denominator turns out to be zero.

2.2 Certain mathematical expressions have been given names because they occur so frequently that it is expedient to do so. For example $a + b$ is called a sum, ab is called a product, $a \div b$ is called a fraction, etc. ab is designated as a double product to distinguish it from a triple product abc, and from a quadruple product $abcd$. Certain polynomials that occur in algebra in the solution of simultaneous linear equations are called determinants. The polynomial $ab - cd$ is a determinant of the second order; $aei + dhc + gbf - ahf - dbi - gec$ is a determinant of the third order. A determinant of the fourth order contains sums and differences of 24 quadruple products, and in general a determinant of the nth order contains $n!$ n-fold products.

A symbol or short-hand notation has been agreed upon which is useful from two important points of view. First: it is much simpler to start with the short-hand notation and a few simple rules and obtain the required determinant. Second: a set of simultaneous equations whose solution results in the ratio of two determinants can be solved much more readily by using the symbolic notation rather than trying to write the determinants directly or obtain them step by step.

A determinant of the nth order is designated by an array of n^2 numbers in n columns and n rows, and this array is enclosed between two vertical lines.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ designates the second order determinant: } ad - bc.† \tag{2.1}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \begin{array}{l} \text{designates the third order determinant:} \\ (aei + dhc + gbf - ahf - dbi - gec). \end{array} \tag{2.2}$$

A more useful convention is to use one letter with two numerical subscripts, as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ designates a third order determinant.} \tag{2.3}$$

In the discussion above we have defined a determinant of the second order and of the third order and have given a symbol to represent each. Before trying to establish a connection between a determinant and its symbol, and before trying to define a determinant of the fourth or higher order, it is desirable to make the following study of permutations.

***2.3 Permutations.** The numbers 1 and 2 can be arranged two different ways:

$$12 \quad \text{and} \quad 21.$$

The numbers 1, 2, and 3 can be arranged six different ways:

$$123, \quad 132, \quad 213, \quad 231, \quad 312, \quad 321.$$

The numbers 1, 2, 3, and 4 can be arranged twenty-four different ways. In general n different integers can be arranged $n!$ different ways. Each of

† When we write

$$\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = (2)(5) - (4)(3) = -2$$

we have on the left the symbol of the determinant, in the middle we have the determinant, and on the right we have the value of the determinant.

Some authors use the term determinant to designate the symbol rather than the polynomial. The nomenclature used in this text is in agreement with that found in L. E. Dickson, "Elementary Theory of Equations," John Wiley and Sons, New York, 1914; J. M. Thomas, "Theory of Equations," McGraw-Hill Book Company, New York, 1938; Garrett Birkhoff and Saunders MacLane, "A Survey of Modern Algebra," The Macmillan Company, New York, 1941; and G. D. Birkhoff, "Determinant," The Encyclopaedia Britannica, Fourteenth Edition, 1929.

these arrangements is known as a permutation. There are two possible permutations of the numbers 1 and 2. There are six possible permutations of the numbers 1, 2, and 3; etc.

*2.4 **Inversions in a Permutation.** Consider a permutation of the numbers 1 to 4:

$$3 \quad 2 \quad 1 \quad 4.$$

Each pair of numbers not in the natural order is called an inversion. In this case 3 comes before 2; therefore, 32 is an inversion. Also 3 comes before 1; therefore, 31 is another inversion. There is one more inversion in this permutation; it is 21. The permutation 13542 contains the following inversions: 32, 54, 52, 42. The permutation 51432 contains the following inversions: 51, 54, 53, 52, 43, 42, 32.

Odd and Even Permutations. If a permutation contains an odd number of inversions it is called an odd permutation. If it contains an even number of inversions it is called an even permutation. The permutation 13542 is even since it contains four inversions, while the permutations 3214 and 51432 are both odd permutations containing three and seven inversions respectively.

*2.5 **Theorem.** The interchanging of any two numbers of a permutation will change the permutation from odd to even, or from even to odd. If we interchange two adjacent numbers the result will be to increase the number of inversions by one if the two numbers interchanged were in the natural order at first. If the two numbers interchanged were not in the natural order, then the interchange will decrease the number of inversions by one. In either case the number would change from odd to even or from even to odd. Suppose now that two numbers not adjacent are interchanged. The result can be analyzed as follows.

Let the numbers that are to be interchanged be designated a and b where a is to the left of b. Suppose there are k numbers between a and b. Interchange a with each of the k numbers to its right and this will put a adjacent to and to the left of b. Now interchange a and b; next interchange b with each of the k numbers to its left. The result will be the same as just interchanging a and b. In going through this process we have interchanged $2k + 1$ pairs of adjacent numbers. Therefore the permutation changed from odd to even or from even to odd $2k + 1$ times. Since $2k + 1$ is an odd number, the result of the interchange of any two numbers is to change the permutation from odd to even or from even to odd.

*2.6 **Theorem.** Of all the permutations of a set of numbers, half are odd and half are even. Suppose we list the odd permutations in one column and the even permutations in another. If we interchange the first two numbers in each of the odd permutations, each will be changed to an even

permutation. Now, since the odd permutations were all different the even permutations produced are all different. Therefore there cannot be more odd permutations than there are even ones.

A similar treatment of the column of even permutations shows that there cannot be more even permutations than there are odd permutations. Therefore there must be the same number of odd permutations as there are even ones.

***2.7 Arranging Numbers in Natural Order.** If the numbers in a permutation are in the natural order, the number of inversions is zero; therefore such a permutation is an even permutation. Any permutation can be arranged in the natural order by interchanging certain pairs of numbers. Since each interchange changes from odd to even or from even to odd it is evident that an odd permutation will require an odd number of interchanges to arrange its numbers in the natural order, and an even permutation will require an even number of such interchanges. For example, to change 3214 to the natural order we need only interchange 1 and 3; therefore 3214 is an odd permutation. To arrange 13542 in the natural order the steps could be: 13245, 12345. Two interchanges indicate 13542 is even. To arrange 51432 we can use the following steps: 15432, 12435, 12345; three steps indicate 51432 is odd.

Conversely, the number of interchanges required to obtain a certain permutation from the natural order will tell whether the permutation is odd or even. To illustrate this we merely go through the steps in the opposite order to what we did above.

2.8 Second Order Determinant. The second order determinant $a_{11}a_{22} - a_{12}a_{21}$ is designated by the symbol

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Note that the first subscript indicates the row and the second subscript indicates the column. a_{12} is in the first row and second column.

The second order determinant $a_{11}a_{22} - a_{12}a_{21}$ contains double products of numbers in the symbol, one number taken from each column and row. When the numbers in each product are arranged so that the first subscripts are in the natural order, if the second subscripts form an even permutation the product is given the sign plus. If the permutation of the second subscripts is odd the product is given the sign minus.

2.9 Third Order Determinant. The third order determinant is indicated by the symbol

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The determinant itself contains triple products taking a number from each row and column. Six such products can be selected.

$$a_{11}a_{22}a_{33}, \qquad a_{12}a_{23}a_{31}, \qquad a_{13}a_{21}a_{32},$$

$$a_{11}a_{23}a_{32}, \qquad a_{12}a_{21}a_{33}, \qquad a_{13}a_{22}a_{31}.$$

The numbers in the above products have been arranged so that the first subscripts are in the natural order. The second subscripts form even permutations in the first three products; therefore, they are given the sign plus. The second subscripts form odd permutations in the last three products; therefore, they have the sign minus. This gives us the relation

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \tag{2.4}$$

2.10 Determinant of Order n. The determinant of order four is defined in a manner similar to the preceding cases. It is indicated by the following symbol:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \tag{2.5}$$

It contains 4^2 numbers which are arranged in $4!$ 4-fold products; the numbers in each product are chosen so that no two are in the same row or column in the above symbol. Each product is given a sign which is determined by the following rule: arrange the numbers so that the first subscripts are in the natural order; if the second subscripts form an odd permutation use the sign minus, if the second subscripts form an even permutation use the sign plus.

To define a determinant of order n we need only change 4 to n in the above definition. In this case the symbol will be

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{2.6}$$

It is worth noting that the symbol is not only a short-hand notation for its determinant but it also simplifies the definition to some extent.

2.11 Definitions. Each of the numbers in the determinant, e.g., a_{11}, a_{23}, etc., is called an **element.** Each of the products in the determinant (double products in second order; triple products in third order, etc.) is called a **term.** The elements in any line running left and right in the

symbol (2.6) constitute a **row.** The elements a_{11}, a_{12}, a_{13} in equation (2.4) make up the first row, etc. The elements in any line running up and down constitute a **column.** The elements a_{12}, a_{22}, a_{32} in equation (2.4) make up the second column, etc. The **order** of a determinant equals the number of rows.

***2.12 Theorem.** If in a determinant the rows are made into columns and the columns are made into rows so that the first row becomes the first column, etc., and the first column becomes the first row, the determinant is unchanged. For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}.$$

Let D be the given determinant. Indicate the determinant formed by the above process by Δ. It is evident from the definition of a determinant that D and Δ contain the same terms. There may be a question about the signs of corresponding terms. Consider a term of D: $a_{1a}a_{2b}a_{3c}\cdots$; this has the sign plus if the permutation $a, b, c\cdots$ is even, minus if odd. If we rearrange the elements in the term $a_{1a}a_{2b}a_{3c}\cdots$ so that the numbers $a, b, c\cdots$ are in the natural order, then the numbers $1, 2, 3\cdots$ will be put in a permutation that is even if $a, b, c\cdots$ is an even permutation and is odd if $a, b, c\cdots$ is an odd permutation; see section 2.7. The corresponding term in Δ is written with $a, b, c\cdots$ in the natural order and is given a plus sign if the numbers $1, 2, 3\cdots$ form an even permutation, and a minus sign if $1, 2, 3\cdots$ form an odd permutation. Therefore every term in Δ has the same sign as the corresponding term in D, and $\Delta = D$.

2.13 Theorem. If two rows (columns) of a determinant are interchanged, the result is to change the sign of the determinant. Let D be the given determinant and let Δ be the determinant that results when we interchange two columns of D. D and Δ contain the same terms. We must investigate the signs of the corresponding terms. Let $a_{1a}a_{2b}a_{3c}\cdots$, be a term of D. This is plus if $a, b, c\cdots$ is an even permutation, minus if $a, b, c\cdots$ is an odd permutation. The corresponding term in Δ is identical except that two of the numbers are interchanged. The term in Δ is therefore plus if $a, b, c\cdots$ is an odd permutation and minus if $a, b, c\cdots$ is an even permutation. Therefore every term of Δ is the negative of the corresponding term in D, and $\Delta = -D$.

If two rows are interchanged we have the following method of proof. Let D be the given determinant; let Δ be the determinant that results when two rows of D are interchanged. Let D_1 be the determinant that results when the rows and columns of D are interchanged so that the first row becomes the first column, etc., and the first column becomes the first

row, etc. Let Δ_1 be the determinant that results when the rows and columns of Δ are interchanged so that the first row becomes the first column, etc., and the first column becomes the first row, etc. Δ_1 is the same as D_1 with two columns interchanged and, by the discussion above, $\Delta_1 = -D_1$. By section 2.12, $D_1 = D$ and $\Delta_1 = \Delta$; therefore $\Delta = -D$ and the theorem is proved.

2.14 Theorem. If the elements of two rows (columns) are alike the determinant equals zero. For example,

$$\begin{vmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{vmatrix} = 0$$

since the second and third columns are alike.

To prove this we need only interchange the two rows (columns) that are alike. By the preceding section this changes the sign of the determinant D. But since the two rows (columns) are alike, interchanging them will make no change in D. Therefore $D = -D$ and therefore $D = 0$.

2.15 Theorem. If each element in a row (column) of a determinant be multiplied by the same factor the result is to multiply the determinant by the same factor. For example,

$$k\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} ka_{11} & a_{12} & a_{13} \\ ka_{21} & a_{22} & a_{23} \\ ka_{31} & a_{32} & a_{33} \end{vmatrix}.$$

Suppose every element in the 4th column (row) of D is multiplied by k. Each term in the determinant Δ which results will be equal to k times the corresponding term of D since each term of Δ contains exactly one element from the 4th column (row). Therefore k is a factor of Δ and when it is removed the quotient will be D or $\Delta = kD$.

2.16 Minors. If we remove from a determinant the row and column containing a specific element we have left a determinant called the minor of that element. For example, in the determinant of equation (2.3),

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \quad \text{is the minor of } a_{33},$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \quad \text{is the minor of } a_{23}.$$

The minor of a_{33} is designated by M_{33}, the minor of a_{12} by M_{12}, etc.

$$\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = M_{22}.$$

2.17 Theorem. A determinant can be expanded by taking the elements of the first column, multiplying each by its minor and, starting with the first element times its minor, assign plus and minus signs alternately. As an illustration,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31}.$$

For a determinant of order n we have

$$D = a_{11}M_{11} - a_{21}M_{21} + a_{31}M_{31} \cdots a_{n1}M_{n1}. \tag{2.7}$$

First we shall show that every term on the right of (2.7) is in D. Then we shall show that the terms have the correct sign in each case. M_{11} contains all possible products of the elements of D chosen one from each row and column, except first row and column. $a_{11}M_{11}$ contains all terms of D that contain a_{11} as a factor. Similarly, $a_{21}M_{21}$ contains all terms of D containing a_{21} as a factor. Finally, the right-hand side of equation (2.7) contains all terms of D containing elements of the first column as factors, and therefore all the terms of D. It is also true that there is no term included on the right-hand side of equation (2.7) that is not a term of D.

Consider a term of $a_{11}M_{11}$,

$$a_{11}a_{2a}a_{3b}a_{4c} \cdots a_{nq}.$$

This is plus if $a, b, c \cdots q$ is an even permutation, minus if an odd permutation. But as a term of D this is plus if $1, a, b \cdots q$ is an even permutation, minus if an odd permutation. It is evident that $1, a, b \cdots q$ and $a, b, \cdots q$ have the same number of inversions and therefore are both odd or both even; therefore, every term in $a_{11}M_{11}$ has the same sign as the corresponding term in D.

Now consider a term of $a_{21}M_{21}$,

$$a_{21}a_{1a}a_{3b}a_{4c} \cdots a_{nr}.$$

This is plus if $a, b, c \cdots r$ is an even permutation, minus if an odd permutation. As a term in D this is plus if $a, 1, b \cdots r$ is an even permutation, minus if an odd permutation. The permutation $a, 1, b, c \cdots r$ contains one inversion (namely $a1$) not found in $a, b, c \cdots r$; otherwise they have the same inversions. Therefore, if $a, 1, b, c \cdots r$ is even, $a, b, c \cdots r$ is odd and, if $a, 1, b, c \cdots r$ is odd, $a, b, c \cdots r$ is even. Therefore each term in $a_{21}M_{21}$ has the opposite sign to the corresponding term in D, and therefore each term in $-a_{21}M_{21}$ has the same sign as the corresponding term in D.

Consider next a term of $a_{31}M_{31}$,

$$a_{31}a_{1a}a_{2b}a_{4c}a_{5d} \cdots a_{ns}.$$

This is plus if $a, b, c, d \cdots s$ is an even permutation, minus if an odd permutation. As a term of D this is plus if $a, b, 1, c, d \cdots s$ is an even permutation, minus if odd. Now the permutation $a, b, 1, c, d \cdots s$ contains two more inversions than $a, b, c, d \cdots s$, namely $a1$ and $b1$. Therefore the permutations are both even or both odd. And therefore each term in $a_{31}M_{31}$ has the same sign as the corresponding term in D.

Since the same type of argument holds for each term on the right of equation (2.7) the theorem is proved.

2.18 Expansion of a Determinant on any Column or Row. A determinant can be expanded on any column (or row) by taking each element of that column (or row), multiplying by its minor, and assigning plus and minus signs as indicated by the following checkerboard diagram.

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

We have shown in the preceding section that if the determinant is expanded on the first column the signs are $+ \quad - \quad + \quad -$ etc. If the determinant is to be expanded on the second column, we can first interchange the first and second columns, which will require a minus sign if the determinant is to remain unchanged. We now expand on the first column using signs $+ \quad - \quad + \quad -$ etc., but since we interchanged two columns the minus sign required to keep the determinant unchanged changes these signs to $- \quad + \quad - \quad +$ etc.

To expand the determinant on the third column interchange the third column successively with the second and first. Two interchanges leave the determinant unchanged and, when we expand on the new first column, we have $+ \quad - \quad + \quad -$ etc. for signs. This argument can be carried through step by step as is done above to show that the above diagram holds for expansion on any column.

If a determinant is to be expanded on any row, we can first interchange rows and columns making the first row the first column, etc., and the first column the first row, etc. This does not change the value of the determinant. We can now use the above diagram for expansion on any column. Since the diagram gives the same sequence of signs for the same row and column, it always gives the correct sign, and the theorem is proved.

2.19 Theorem. If each element in any row (or in any column) is equal to zero, the determinant equals zero. To show this we need only expand the determinant on the row (or column) containing the zeros.

$$\begin{vmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} = -0\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + 0\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - 0\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{33} \end{vmatrix}$$

$$= 0. \tag{2.8}$$

2.20 Numerical Examples. A determinant can be evaluated readily by the method of the preceding section if several of its elements are equal to zero.

$$\begin{vmatrix} 4 & 0 & 5 \\ 2 & 6 & 0 \\ 0 & 2 & 3 \end{vmatrix} = 4\begin{vmatrix} 6 & 0 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & 5 \\ 2 & 3 \end{vmatrix}$$

$$\begin{aligned} &= 4(6 \times 3) - 2(-2 \times 5) \\ &= 72 \qquad\quad + 20 \\ &= 92. \end{aligned}$$

$$\begin{vmatrix} 4 & 0 & 3 & 5 \\ 3 & 8 & 0 & 0 \\ 2 & 5 & 0 & 3 \\ 0 & 6 & 7 & 7 \end{vmatrix} = -3\begin{vmatrix} 0 & 3 & 5 \\ 5 & 0 & 3 \\ 6 & 7 & 7 \end{vmatrix} + 8\begin{vmatrix} 4 & 3 & 5 \\ 2 & 0 & 3 \\ 0 & 7 & 7 \end{vmatrix}$$

$$= 3 \cdot 5\begin{vmatrix} 3 & 5 \\ 7 & 7 \end{vmatrix} + 3 \cdot 3\begin{vmatrix} 0 & 3 \\ 6 & 7 \end{vmatrix} - 8 \cdot 2\begin{vmatrix} 3 & 5 \\ 7 & 7 \end{vmatrix}$$

$$- 8 \cdot 3\begin{vmatrix} 4 & 3 \\ 0 & 7 \end{vmatrix}$$

$$\begin{aligned} &= 15(21 - 35) + 9(-18) - 16(21 - 35) - 24(28) \\ &= -210 - 162 + 224 - 672 \\ &= -820. \end{aligned}$$

If the determinant to be evaluated contains few zero elements it is frequently better to reduce the order of the determinant by the method developed in the following sections.

2.21 Addition of Determinants. If two determinants are identical except possibly for the elements in the sth column (or row), their sum is equal to a determinant identical with the given determinants except for the elements in the sth column (or row). The elements in the sth column (or row) of the sum equals the sums of the corresponding elements of the sth columns (or rows) of the given determinants.

Assume that the given determinants are identical except for the elements in the first column. Let the elements in the first column of the first determinant be a_{11}, a_{21}, $a_{31} \cdots a_{n1}$; and let the elements in the first column of

the second determinant be $b_{11}, b_{21} \cdots b_{n1}$. The sum of the first two determinants expanded on their first columns is in terms of the elements in the first columns and their minors:

$$D_1 + D_2 = a_{11}M_{11} - a_{21}M_{21} + \cdots a_{n1}M_{n1} + b_{11}M_{11} \\ - b_{21}M_{21} + \cdots b_{n1}M_{n1}. \tag{2.9}$$

The minor of a_{11} is identical to the minor of b_{11} since the two determinants differ only in the first column. The same is true for the other minors; the minor of a_{k1} is exactly the same as the minor of b_{k1} for any value of k. For this reason we find M_{11} multiplied by both a_{11} and b_{11} in equation (2.9).

If we rearrange the right-hand side of equation (2.9) we have

$$D_1 + D_2 = (a_{11} + b_{11})M_{11} - (a_{21} + b_{21})M_{21} + \cdots (a_{n1} + b_{n1})M_{n1}. \tag{2.10}$$

The right-hand side of equation (2.10) is a determinant, call it D, identical to the given determinants except for the elements of the first column, and the elements of the first column of D are equal to the sums of the corresponding elements of the first columns of the two given determinants. This proves the theorem for the special case of the first column.

If the determinants are alike except for the sth columns, the sth columns can be interchanged with the first columns which will change the signs of all three determinants. The discussion above will show that the negatives of the determinants satisfy the theorem

$$-D_1 - D_2 = -D.$$

Therefore, the determinants themselves do, also.

$$D_1 + D_2 = D.$$

The theorem can be shown to obtain for rows by merely interchanging the rows and columns.

2.22 Theorem. If a constant times the elements of a row (or column) are added to the corresponding elements of any other row (or column), the determinant is left unchanged.

The preceding theorem shows that the following equation is true.

$$D = \begin{vmatrix} a_{11} + ka_{12} & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} + ka_{22} & a_{22} & \cdot & & \cdot \\ a_{31} + ka_{32} & a_{32} & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & \cdot & \\ a_{31} & a_{32} & a_{33} & \cdot & \\ \cdot & \cdot & \cdot & & \end{vmatrix} + \begin{vmatrix} ka_{12} & a_{12} & a_{13} & \cdot & \cdot \\ ka_{22} & a_{22} & a_{23} & \cdot & \cdot \\ ka_{32} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \end{vmatrix}. \tag{2.11}$$

Now, using section 2.15, we have

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} + k \begin{vmatrix} a_{12} & a_{12} & a_{13} & \cdot & \cdot \\ a_{22} & a_{22} & a_{23} & \cdot & \cdot \\ a_{32} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix}. \tag{2.12}$$

But, according to section 2.14 the second determinant on the right of equation (2.12) is zero. Therefore,

$$\begin{vmatrix} a_{11} + ka_{12} & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} + ka_{22} & a_{22} & a_{23} & \cdot & \cdot \\ a_{31} + ka_{32} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \end{vmatrix}. \tag{2.13}$$

The theorem can now be shown to be true for columns other than the first two by interchanging the columns to bring them into the first and second place. The validity of the theorem for rows follows from the fact that columns and rows can be interchanged without changing the value of the determinant.

It is recommended that the student review section 2.15 at this point. This section and section 2.15 describe two distinct properties of determinants that must not be confused.

2.23 Theorem. If the element in the rth row and the sth column is not equal to zero, then all the other elements in the rth row and sth column can be made equal to zero without changing the value of the determinant.

In the preceding section, if $a_{12} \neq 0$, let

$$k = -\frac{a_{11}}{a_{12}}.$$

This gives zero in the upper left-hand corner.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot \\ a_{31} & a_{32} & a_{33} & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} = \begin{vmatrix} 0 & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} - \dfrac{a_{11}a_{22}}{a_{12}} & a_{22} & a_{23} & \cdot & \cdot \\ a_{31} - \dfrac{a_{11}a_{32}}{a_{12}} & a_{32} & a_{33} & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix}. \tag{2.14}$$

If we now add $-a_{13}/a_{12}$ times the elements of the second column to the

elements of the third column, we get a zero at the top of the third column.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} = \begin{vmatrix} 0 & a_{12} & 0 & \cdot \\ a_{21} - \dfrac{a_{11}a_{22}}{a_{12}} & a_{22} & a_{23} - \dfrac{a_{13}a_{22}}{a_{12}} & \cdot \\ a_{31} - \dfrac{a_{11}a_{32}}{a_{12}} & a_{32} & a_{33} - \dfrac{a_{13}a_{32}}{a_{12}} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{2.15}$$

This process can be continued until every element in the first row but the second one becomes zero. Then by a similar process every element but the first one in the second column can be made zero.

2.24 Reduction of the Order of a Determinant. If every element but one of any row (or column) equals zero, the determinant equals that element times its minor with the sign plus or minus assigned according to the checkerboard array:

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{matrix}$$

This follows from section 2.18. We have therefore reduced the order of the determinant by one. As an illustration,

$$\begin{vmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & 0 & a_{43} & a_{44} \end{vmatrix} = -a_{32} \begin{vmatrix} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \\ a_{41} & a_{43} & a_{44} \end{vmatrix}. \tag{2.16}$$

We see from this and the preceding section that it is always possible to reduce the order of a determinant. If a given numerical determinant does not contain many zero elements it is sometimes better to reduce the determinant by the present method. A determinant with many zeros can usually be evaluated more easily by expanding it as described in section 2.18.

2.25 Diagonal Determinant. Theorem. If every element on one side of the diagonal from the upper left-hand corner equals zero, the elements on the other side of the diagonal can be replaced by zeros without changing the value of the determinant. The value of the determinant is equal to the product of the elements in the diagonal.

Assume that all the elements below the leading diagonal are equal to zero. Now consider any term other than the product of the elements of

the leading diagonal. If the term does not contain a_{11}, it must contain one of the other elements in the first column; since all the other elements in the first column are equal to zero any term not containing a_{11} will equal zero. Now consider a term containing a_{11} but not a_{22}. The term cannot contain a_{12} since that is in the same row as a_{11}; therefore, it must contain some element of the second column other than a_{12} and a_{22}. Since every element in the second column except a_{12} and a_{22} is zero by hypothesis every term that does not contain a_{11} and a_{22} will be zero. This argument is continued for every column but the last, and we find every term except the product of the elements in the leading diagonal is equal to zero, and therefore the determinant is equal to the product of the elements in the leading diagonal.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot \\ 0 & a_{22} & a_{23} & \cdot & \cdot \\ 0 & 0 & a_{33} & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} = a_{11}a_{22}a_{33}\cdots. \tag{2.17}$$

If the determinant equals the product of the elements in the diagonal when the elements on one side of the diagonal are all zero, then its value is independent of the values of the elements on the other side of the diagonal and they may be given any value we please. Therefore they may be made zero.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdot & \cdot \\ 0 & a_{22} & a_{23} & a_{24} & \cdot & \cdot \\ 0 & 0 & a_{33} & a_{34} & \cdot & \cdot \\ 0 & 0 & 0 & a_{44} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 & 0 & \cdot & \cdot \\ 0 & a_{22} & 0 & 0 & \cdot & \cdot \\ 0 & 0 & a_{33} & 0 & \cdot & \cdot \\ 0 & 0 & 0 & a_{44} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}. \tag{2.18}$$

2.26 Theorem. Any determinant not equal to zero can be made into a diagonal determinant. This follows directly from section 2.23. In case the determinant is equal to zero see section 2.34.

2.27 Cramer's Rule. Given a set of n linear equations in n unknowns.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n &= k_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots a_{nn}x_n &= k_n. \end{aligned} \tag{2.19}$$

Let D be the nth order determinant formed by the coefficients of the unknowns as they are written in the equations above.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{2.20}$$

Let D_1 be the determinant obtained when the elements of the first column of D are replaced by the constants on the right of equations (2.19).

$$D_1 = \begin{vmatrix} k_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ k_2 & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ k_n & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{2.21}$$

Let D_2 be the determinant obtained when the elements in the second column of D are replaced by the constants on the right of the equations similar to the way D_1 was obtained. Define $D_3, D_4, \cdots D_n$ in a similar manner.

Cramer's rule states: If $D \neq 0$,

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}, \quad \cdots \quad x_n = \frac{D_n}{D}.$$

The relation $x_1 = D_1/D$ can be derived as follows. Application of section 2.15 gives

$$x_1 D = \begin{vmatrix} x_1a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ x_1a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ x_1a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{2.22}$$

Add x_2 times the elements of the second column to the corresponding elements of the first column:

$$x_1 D = \begin{vmatrix} x_1a_{11} + x_2a_{12} & a_{12} & a_{13} & \cdots & a_{1n} \\ x_1a_{21} + x_2a_{22} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ x_1a_{n1} + x_2a_{n2} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}. \tag{2.23}$$

Continue by adding x_3 times the elements of the third column to the corresponding elements of the first, then add x_4 times the elements of the fourth column to the corresponding elements of the first, etc. We finally add x_n times the elements of the last column to the corresponding elements of the first column.

The elements in the first column will now be identical with the left-hand sides of the given equations. We can therefore replace the elements in the first column by the right-hand sides of the given equations. When we do this we get D_1; therefore,

$$x_1 D = D_1. \tag{2.24}$$

If $D \neq 0$ we can divide equation (2.24) by D giving

$$x_1 = \frac{D_1}{D}. \tag{2.25}$$

The other relations can be found by similar reasoning.

2.28 Matrix. In order to discuss the case omitted in the preceding section, when $D = 0$, and to know what to do when the number of equations is not the same as the number of unknowns, it is convenient to use the concept of a matrix.

A set of mn quantities arranged in a rectangular array having m rows and n columns is called a matrix. The rectangular array is usually enclosed between two pairs of parallel lines. For example,

$$\begin{Vmatrix} 2 & 5 & 8 \\ 1 & 2 & 5 \end{Vmatrix} \qquad \begin{Vmatrix} 3 & 4 & 0 \\ 1 & 5 & 3 \\ 2 & 9 & 6 \end{Vmatrix}$$

are matrices.

A matrix does not have a numerical value in the same sense that a determinant does. However, by crossing out certain rows and columns, we can get square arrays of numbers which we may consider determinants. Any determinant which can be obtained by this method is a **determinant of the matrix.** The matrix to the left above contains 3 second order determinants and 6 first order determinants. The matrix on the right contains 1 third order determinant, 9 second order determinants, and 9 first order determinants.

2.29 Rank of a Matrix. The rank of a matrix is said to be r if it contains at least one non-zero determinant of order r, and all its determinants of order greater than r are zero.

The rank of each of the following matrices is 2.

$$\begin{Vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{Vmatrix} \qquad \begin{Vmatrix} 2 & 4 & 7 \\ 1 & 1 & 3 \\ 3 & 5 & 10 \end{Vmatrix}$$

2.30 Rank of a Determinant. The elements in a determinant may be considered as making up a square matrix. This matrix is called **the matrix of the determinant.** And conversely the highest order determinant of a square matrix is called **the determinant of the matrix.** The rank of a determinant is the same as the rank of the matrix of the determinant. The rank of the following determinant is two.

$$\begin{vmatrix} 2 & 4 & 7 \\ 1 & 1 & 3 \\ 3 & 5 & 10 \end{vmatrix}$$

The rank of the following determinant is three.

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix}$$

2.31 Elementary Transformation of a Matrix. The following types of transformations are called elementary transformations:

(1) Interchange of two columns (or rows).

(2) Multiplication of each element of a column (or row) by the same non-zero constant.

(3) The addition to the elements of a column (or row) of the corresponding elements of any other column (or row) each multiplied by the same constant.

Note that, if it is possible to pass from the matrix M to the matrix N by any one of the elementary transformations, it is possible to pass from N to M by one of the elementary transformations.

2.32 Equivalent Matrices. If it is possible to pass from one matrix to another by a finite number of elementary transformations, the matrices are said to be equivalent.

2.33 Theorem. If two matrices are equivalent they are of the same rank. A transformation of the first type merely changes the sign of a determinant by the theorem of section 2.13. A transformation of the second type multiplies a determinant by a constant not zero by the theorem of section 2.15. Therefore the first two types of transformations do not affect the vanishing of a determinant in the matrix and therefore do not affect the rank of the matrix.

Suppose we change matrix M into matrix N by adding k times the elements of the tth row to the corresponding elements of the sth row. If M is of rank r every $(r + 1)$-rowed determinant of M is zero. Any $(r + 1)$-rowed determinant of N which does not contain the sth row of M is unchanged and therefore equals zero. Any $(r + 1)$-rowed determinant of N containing both the t and sth row of M is unchanged by section 2.22 and

therefore equals zero. Any $(r+1)$-rowed determinant of N containing the sth row of M and not the tth row can be written $P + kQ$, by section 2.21, where P and Q are $(r+1)$-rowed determinants of M and are therefore equal to zero. We have thus shown that every $(r+1)$-rowed determinant of N is equal to zero. Therefore the rank of N is not greater than the rank of M. Interchanging M and N in the above discussion shows that the rank of M cannot be greater than the rank of N. Therefore M and N both have the same rank.

2.34 Theorem. Any matrix of rank r can be put into the following form by means of elementary transformations: the element in the sth row and sth column equals unity where $s \leqq r$, and every other element equals zero. To do this we need only rearrange the columns and rows to obtain a nonvanishing r-order determinant in the upper left-hand corner. This determinant can be changed to a diagonal determinant by section 2.26 which makes use of the first and third elementary transformation. Using the second elementary transformation we can make each element in the diagonal equal to unity. It is now a simple matter to make every element of the matrix in the first r columns, below the rth row, equal to zero using the third transformation. Similarly we make every element in the first r-rows to the right of the rth column equal to zero. If any element below the rth row and to the right of the rth column is not equal to zero, by means of the first transformation we can move it to the $(r+1)$ row and $(r+1)$ column and we have a determinant of order $(r+1)$ not equal to zero. Therefore we have shown that we can always obtain the above form.

2.35 Procedure for Finding the Rank of a Matrix. Finding the rank of a matrix tends to be a rather laborious process. For example, a square matrix having five rows contains 1 fifth order, 25 fourth order, 100 third order, 100 second order, and 25 first order determinants. To find that a given matrix having five rows and five columns is of rank two would require us to evaluate at least the 100 third order determinants unless we could find some short cuts.

If we go through some of the steps outlined in the preceding section we can do away with unnecessary and duplicate steps. To illustrate the recommended procedure we include the following examples.

Example 1.

$$\begin{Vmatrix} 5 & 2 & 2 & 3 & 4 \\ 8 & 3 & 4 & 4 & 7 \\ 9 & 8 & 6 & 1 & 5 \end{Vmatrix}.$$

Subtract the first row from the second row.

$$\begin{Vmatrix} 5 & 2 & 2 & 3 & 4 \\ 3 & 1 & 2 & 1 & 3 \\ 9 & 8 & 6 & 1 & 5 \end{Vmatrix}.$$

Subtract 2 times the second row from the first and subtract 8 times the second row from the third.

$$\left\|\begin{matrix} -1 & 0 & -2 & 1 & -2 \\ 3 & 1 & 2 & 1 & 3 \\ -15 & 0 & -10 & -7 & -19 \end{matrix}\right\|.$$

Interchange the first and second rows.

$$\left\|\begin{matrix} 3 & 1 & 2 & 1 & 3 \\ -1 & 0 & -2 & 1 & -2 \\ -15 & 0 & -10 & -7 & -19 \end{matrix}\right\|.$$

Interchange the first and second columns.

$$\left\|\begin{matrix} 1 & 3 & 2 & 1 & 3 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & -15 & -10 & -7 & -19 \end{matrix}\right\|.$$

Replace the elements in the first row other than the first by zeros.

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & -15 & -10 & -7 & -19 \end{matrix}\right\|.$$

This is the same as subtracting 3 times the first column from the second column, etc.

Now subtract 15 times the second row from the third row.

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 1 & -2 \\ 0 & 0 & 20 & -22 & 11 \end{matrix}\right\|.$$

Since the determinant of the first three columns equals $-20 \neq 0$ the matrix is of rank three.

Example 2.

$$\left\|\begin{matrix} 1 & 4 & 3 & 7 & 6 & -2 \\ 7 & 1 & 2 & 6 & -4 & 2 \\ 3 & -6 & 3 & -5 & 2 & 7 \\ 11 & -1 & 8 & 8 & 4 & 7 \end{matrix}\right\|.$$

Subtract 4 times the second row from the first row, add 6 times the second row to the third row, and add the second row to the fourth row.

$$\left\|\begin{matrix} -27 & 0 & -5 & -17 & 22 & -10 \\ 7 & 1 & 2 & 6 & -4 & 2 \\ 45 & 0 & 15 & 31 & -22 & 19 \\ 18 & 0 & 10 & 14 & 0 & 9 \end{matrix}\right\|.$$

Interchange the first two columns and interchange the first two rows; we can replace the elements 7, 2, 6, -4, 2 in the new first row by zeros.

$$\left\|\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -27 & -5 & -17 & +22 & -10 \\ 0 & 45 & 15 & 31 & -22 & 19 \\ 0 & 18 & 10 & 14 & 0 & 9 \end{matrix}\right\|.$$

Add the third row to the second row.

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 18 & 10 & 14 & 0 & +9 \\ 0 & 45 & 15 & 31 & -22 & 19 \\ 0 & 18 & 10 & 14 & 0 & 9 \end{Vmatrix}.$$

Interchange the second and third rows and interchange the second and fifth columns; replace the elements 45, 15, 31, 19 in the old third row by zeros.

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 14 & 18 & 9 \\ 0 & 0 & 10 & 14 & 18 & 9 \end{Vmatrix}.$$

Subtract the third row from the fourth row.

$$\begin{Vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -22 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 14 & 18 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

Therefore the rank is three.

2.36 Definitions. If we have a set of m linear equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= k_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots \qquad\qquad + a_{2n}x_n &= k_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots \qquad\qquad + a_{mn}x_n &= k_m. \end{aligned} \tag{2.26}$$

The matrix formed by taking the coefficients of the unknowns as they stand in the equations,

$$\begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{Vmatrix}$$

is called **the matrix of the system.**

If we augment the matrix of the system by including the constants which appear on the right of the given equations, in an extra column we have what is called **the augmented matrix.**

The augmented matrix for the above equations is

$$\begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & k_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & k_2 \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ \cdot & \cdot & & \cdot & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & k_m \end{Vmatrix}.$$

2.37 Theorem. The rank of the augmented matrix is not less than the rank of the matrix of the system. This must be so since every determinant of the matrix of the system is a determinant of the augmented matrix.

2.38 Theorem. If the rank of the augmented matrix is greater than the rank of the matrix of the system, the equations are not consistent. This can be shown as follows. Assume the rank of the matrix of the system is r. Multiply the constant term in each equation by X (or any letter which does not already appear in the equations) and rewrite the equations so that the term containing X comes first. It is evident by the way X was introduced that $X = 1$ to satisfy the equations. Arrange the equations and other unknowns so that an $(r + 1)$-rowed determinant not equal to zero is in the upper left corner of the matrix of the coefficients. If we now take the first $(r + 1)$ equations and transpose the $(n - r)$ last unknowns to the right of the equals sign, we can solve for X as a function of the $n - r$ unknowns using Cramer's rule. We get a denominator not equal to zero; and in the numerator we have $(r + 1)$-rowed determinants of the matrix of the system (none of these contain the constants which are now coefficients of X) which are all zero since the rank of the matrix of the system is r. Therefore, to satisfy the equations, $X = 0$. X cannot equal zero and unity; therefore the equations are inconsistent.

As an example consider the equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= k_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= k_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= k_3. \end{aligned} \tag{2.27}$$

Assume the matrix of the system of rank 2. Therefore, we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0. \tag{2.28}$$

Assume the augmented matrix of rank 3. In particular assume

$$\begin{vmatrix} k_1 & a_{11} & a_{12} \\ k_2 & a_{21} & a_{22} \\ k_3 & a_{31} & a_{32} \end{vmatrix} = \Delta \neq 0. \tag{2.29}$$

Rewrite the given equations

$$\begin{aligned} k_1x - a_{11}x_1 - a_{12}x_2 &= a_{13}x_3, \\ k_2x - a_{21}x_1 - a_{22}x_2 &= a_{23}x_3, \\ k_3x - a_{31}x_1 - a_{32}x_2 &= a_{33}x_3. \end{aligned} \tag{2.30}$$

It is evident the way these equations were obtained that $x = 1$. Now, since $\Delta \neq 0$, we can use Cramer's rule to solve equations (2.30) for x.

$$x = \frac{\begin{vmatrix} a_{13}x_3 & -a_{11} & -a_{12} \\ a_{23}x_3 & -a_{21} & -a_{22} \\ a_{33}x_3 & -a_{31} & -a_{32} \end{vmatrix}}{\begin{vmatrix} k_1 & -a_{11} & -a_{12} \\ k_2 & -a_{21} & -a_{22} \\ k_3 & -a_{31} & -a_{32} \end{vmatrix}} = x_3 \frac{\begin{vmatrix} a_{13} & a_{11} & a_{12} \\ a_{23} & a_{21} & a_{22} \\ a_{33} & a_{31} & a_{32} \end{vmatrix}}{\Delta} = 0.$$

Since we agreed that $x = 1$ and now have $x = 0$, the equations are not consistent.

2.39 Theorem. If the rank of the augmented matrix equals the rank of the matrix of the system the equations are consistent. This can be shown as follows. Assume the rank of both matrices is r; then every $(r + 1)$-rowed determinant of the augmented matrix equals zero. Arrange the equations and unknowns so that the r-rowed determinant in the upper left corner of the matrix of the system is not zero. Transpose the other $(n - r)$ unknowns to the right of the equals sign. Using Cramer's rule on the first r equations, we can solve for the r unknowns on the left as functions of the constants and the $(n - r)$ unknowns on the right.

If we substitute the set of values of the r unknowns into any of the original equations we get a sum of $(r + 1)$-rowed determinants to be equal to zero if the equation is satisfied. Each of the $(r + 1)$-rowed determinants is zero because it has two rows identical or because it is an $(r + 1)$-rowed determinant of a matrix of rank r; therefore, the equation is satisfied and the set of equations is consistent.

As an illustration consider the equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= k_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= k_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= k_3. \end{aligned} \tag{2.32}$$

where the augmented matrix is of rank 2, and assume in particular

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = D \neq 0. \tag{2.33}$$

If we take the first two equations and write them

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= k_1 - a_{13}x_3, \\ a_{21}x_1 + a_{22}x_2 &= k_2 - a_{23}x_3. \end{aligned} \tag{2.34}$$

We can solve for x_1 and x_2 using Cramer's rule as follows:

$$x_1 = \frac{\begin{vmatrix} k_1 - a_{13}x_3 & a_{12} \\ k_2 - a_{23}x_3 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix} - x_3 \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix}}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & k_1 - a_{13}x_3 \\ a_{21} & k_2 - a_{23}x_3 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix} - x_3 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}}{D}$$

Now, if we substitute these in the sth equation, we have

$$a_{s1} \frac{\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix} - x_3 \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix}}{D} + a_{s2} \frac{\begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix} - x_3 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}}{D} + a_{s3}x_3 - k_s.$$

If the values given above for x_1 and x_2 satisfy the sth equation this should be zero. This result can be written

$$\frac{1}{D}\left[a_{s1}\begin{vmatrix} k_1 & a_{12} \\ k_2 & a_{22} \end{vmatrix} - a_{s1}x_3 \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix} + a_{s2} \begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix}\right.$$
$$\left. - a_{s2}x_3 \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{s3}x_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - k_s \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right]$$
$$= \frac{1}{D}\left[- a_{s1} \begin{vmatrix} a_{12} & k_1 \\ a_{22} & k_2 \end{vmatrix} + a_{s2} \begin{vmatrix} a_{11} & k_1 \\ a_{21} & k_2 \end{vmatrix} - k_s \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix}\right.$$
$$\left. + x_3a_{s1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} - x_3a_{s2} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + x_3a_{s3} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\right]$$
$$= \frac{1}{D}\left[- \begin{vmatrix} a_{s1} & a_{s2} & k_s \\ a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{vmatrix} + x_3 \begin{vmatrix} a_{s1} & a_{s2} & a_{s3} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix} \right].$$

If $s = 1$ or 2, two rows are identical and therefore this is zero. If $s = 3$ we have 3-rowed determinants of the augmented matrix which was given as of rank 2; therefore in this case, too, the determinants are zero. Consequently the values given above for x_1 and x_2 satisfy all the equations.

2.40 Theorem. A necessary and sufficient condition that a system of m linear equations in n unknowns be consistent is that the matrix of the system have the same rank as the augmented matrix. If the rank of both matrices is r, then any r unknowns we choose can be expressed as functions

of the constants and the remaining $n - r$ unknowns, provided that the matrix of the coefficients of the chosen unknowns is of rank r.

The theorems of section 2.37 and 2.38 show that the above is a necessary condition. The theorem of section 2.39 shows that it is a sufficient condition. In the proof of the theorem of section 2.39 it was shown that, when the rank of the augmented matrix and the rank of the matrix of the system are both equal to r, r unknowns can be expressed in terms of the constants and the remaining $n - r$ unknowns, provided that the matrix of the coefficient of the chosen unknowns is of rank r.

2.41 Theorem. A necessary condition that a set of $n + 1$ linear equations in n unknowns be consistent is that the determinant of the augmented matrix be zero. (Note that this is a necessary condition but not a sufficient condition.)

The augmented matrix has $n + 1$ columns and $n + 1$ rows. If its determinant (see section 2.30) is not equal to zero the rank of the augmented matrix will be $n + 1$. The matrix of the system contains n columns and $n + 1$ rows. Therefore the rank of the matrix of the system cannot be greater than n. Therefore, if the equations are consistent, the rank of the augmented matrix must not be greater than n; then the determinant of the augmented matrix must be zero.

2.42 Homogeneous Equations. The m linear equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots& \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{2.34}$$

are called **homogeneous equations.** They differ from equations (2.26) in section 2.36 in that all the constants are zero.

If there are as many equations as there are unknowns ($m = n$), the determinant of the matrix of the system is called **the determinant of the coefficients.**

2.43 Theorem. Any set of homogeneous linear equations is consistent.

The augmented matrix of a set of homogeneous linear equations differs from the matrix of the system only in the presence of an additional column every element of which is zero. Therefore the rank of both matrices is the same, and the equations are consistent. Indeed it is apparent that $x_1 = x_2 = x_3 = \cdots = x_n = 0$ is a solution of any set of homogeneous equations.

2.44 Theorem. A necessary and sufficient condition that a set of n homogeneous equations in n unknowns have a solution other than the trivial one, where each unknown equals zero, is that the determinant of the coefficients vanish.

If the determinant of the coefficients equals $D \neq 0$ we can solve for the n unknowns using Cramer's rule. In each case we have D in the denominator and a determinant containing a column of zeros in the numerator. Therefore each unknown equals zero. This shows that it is necessary that $D = 0$, if there is to be a solution other than the one where each unknown is zero.

If the determinant of the coefficients equals zero, its rank is less than n; designate it by $r < n$. By the theorem of section 2.40 we can solve for r unknowns in terms of $n - r$ unknowns provided that the matrix of the coefficients of the r unknowns is of rank r. We may assign any values we please to the $n - r$ unknowns. By assigning the $n - r$ unknowns values other than zero we obtain a set of values of the unknowns, not all equal to zero, which satisfies the given equations.

As an example consider the equations

$$\begin{aligned} x + 3y - 4z &= 0, \\ 2x - 4y + 5z &= 0, \\ x + y - z &= 0. \end{aligned} \tag{2.35}$$

The determinant of the coefficients is

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & -4 & 5 \\ 1 & 1 & -1 \end{vmatrix} = -4,$$

$$x = \frac{\begin{vmatrix} 0 & 3 & -4 \\ 0 & -4 & 5 \\ 0 & 1 & -1 \end{vmatrix}}{-4} = 0, \quad y = \frac{\begin{vmatrix} 1 & 0 & -4 \\ 2 & 0 & 5 \\ 1 & 0 & -1 \end{vmatrix}}{-4} = 0,$$

$$z = \frac{\begin{vmatrix} 1 & 3 & 0 \\ 2 & -4 & 0 \\ 1 & 1 & 0 \end{vmatrix}}{-4} = 0.$$

Since the determinant of the coefficients above is not zero, $x = y = z = 0$ is the only solution that can be found.

As a second example consider the equations

$$\begin{aligned} x + 3y - 4z &= 0, \\ 2x - 4y + 5z &= 0, \\ 3x - y + z &= 0. \end{aligned} \tag{2.36}$$

The determinant of the coefficients is

$$\begin{vmatrix} 1 & 3 & -4 \\ 2 & -4 & 5 \\ 3 & -1 & 1 \end{vmatrix} = 0.$$

Solve the following sets of simultaneous equations:

30. $\begin{aligned} 2x + 3y &= 1 \\ 3x - 5y &= 2 \end{aligned}$

31. $\begin{aligned} x + 2y &= 5 \\ 2x + 3y &= 1 \end{aligned}$

32. $\begin{aligned} x + 2y &= 2 \\ x + 2z &= 3 \\ 2y + z &= 5 \end{aligned}$

33. $\begin{aligned} 2x + 3y + z &= 0 \\ 3x - 2y - 3z &= 1 \\ x - y + 7z &= 2 \end{aligned}$

34. $\begin{aligned} 2x + y &= 1 \\ 3y + z &= 2 \\ 2z + w &= 3 \\ 5x + w &= 4 \end{aligned}$

35. Specify the number of second order determinants in the matrix:

$$\begin{Vmatrix} 1 & 5 & 8 & 1 \\ 2 & 0 & 7 & 0 \\ 0 & 2 & 3 & 5 \end{Vmatrix}$$

36. Specify the number of third order determinants in the matrix of problem 35.

37. Specify the number of determinants of each order from one to six in a square matrix having six rows and six columns.

38. Specify the number of determinants of each order from one to five in a matrix having five rows and six columns.

39. Determine the rank of the matrix of problem 35.

40. Determine the rank of the determinant in problem 26.

41. Determine the rank of the determinant in problem 29.

42. Determine the rank of the following matrix:

$$\begin{Vmatrix} 2 & 1 & 3 & 7 & 9 & 11 \\ 4 & 1 & 2 & 1 & 1 & 2 \\ 6 & 2 & 7 & 1 & 5 & 9 \\ 6 & 2 & 5 & 8 & 10 & 13 \\ 10 & 3 & 9 & 2 & 6 & 11 \end{Vmatrix}$$

Discuss the following systems of equations:

43. $\begin{aligned} x + 2y + 3z &= 5 \\ 2x + y - 7z &= 2 \\ 3x + 3y - 4z &= 1 \end{aligned}$

44. $\begin{aligned} 5x - 2y + 3z &= 2 \\ 3x + y + 4z &= -1 \\ 4x - 3y + z &= 3 \end{aligned}$

45. $\begin{aligned} x + 2y - 5z &= 3 \\ 2x + y - 3z &= 2 \\ 3x + 3y - 8z &= 5 \\ 2x + 4y - 10z &= 6 \\ 3y - 7z &= 4 \end{aligned}$

46. $\begin{aligned} x + 2y + 2z - w &= 3 \\ 2x - y - z + 7w &= 4 \\ 3x + y + z + 6w &= 6 \end{aligned}$

47. $\begin{aligned} x + 2y - 3z &= 0 \\ 4x + 2y - 8z &= 0 \\ 5x + 4y - 11z &= 0 \end{aligned}$

48. $\begin{aligned} 3x - 2y + 3z &= 0 \\ 4x + 2y + z &= 0 \\ x + 2y - 8z &= 0 \end{aligned}$

49. $\begin{aligned} 2x - y + 3z + w &= 0 \\ 4x + 3y - 8z + 2w &= 0 \\ 6x + 2y - 5z + 3w &= 0 \\ 8x + y - 2z + 4w &= 0 \end{aligned}$

8. Repeat problem 7 on the permutations in problem 3.

Evaluate the following determinants:

9. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

10. $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$

11. $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$

12. $\begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix}$

13. $\begin{vmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 6 \end{vmatrix}$

14. $\begin{vmatrix} 1 & 0 & 5 \\ 2 & 2 & 6 \\ 0 & 1 & 0 \end{vmatrix}$

15. $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 5 & 0 & 2 \end{vmatrix}$

16. $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 5 & 0 & 2 & 1 \\ 0 & 8 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}$

17. Evaluate the minor of each element in the following determinants:

$$\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \qquad \begin{vmatrix} 2 & 4 & 9 \\ 1 & 5 & 3 \\ 0 & 7 & 6 \end{vmatrix}$$

Check the following equations by evaluating the determinants:

18. $\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 5 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 1 & 7 \\ 2 & 3 & 0 \end{vmatrix}$

19. $\begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 5 & 7 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 5 & 0 & 7 \end{vmatrix}$

20. $\begin{vmatrix} 1 & 0 & 6 \\ 2 & 3 & 5 \\ 2 & 3 & 5 \end{vmatrix} = 0$

21. $2\begin{vmatrix} 1 & 0 & 2 \\ 2 & 5 & 0 \\ 0 & 3 & 7 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 2 \\ 4 & 5 & 0 \\ 0 & 3 & 7 \end{vmatrix}$

Make the following determinants into diagonal determinants:

22. $\begin{vmatrix} 0 & 3 \\ 2 & 5 \end{vmatrix}$

23. $\begin{vmatrix} 3 & 7 \\ 5 & 11 \end{vmatrix}$

24. $\begin{vmatrix} 0 & 5 & 3 \\ 7 & 9 & 2 \\ 4 & 0 & 6 \end{vmatrix}$

25. $\begin{vmatrix} 2 & 3 & 6 \\ 8 & 7 & 2 \\ 1 & 5 & 4 \end{vmatrix}$

Evaluate the following determinants:

26. $\begin{vmatrix} 5 & 1 & 2 \\ 3 & 2 & 8 \\ 1 & 4 & 1 \end{vmatrix}$

27. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

28. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{vmatrix}$

29. $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{vmatrix}$

Solve the following sets of simultaneous equations:

30. $2x + 3y = 1$
$3x - 5y = 2$

31. $x + 2y = 5$
$2x + 3y = 1$

32. $x + 2y = 2$
$x + 2z = 3$
$2y + z = 5$

33. $2x + 3y + z = 0$
$3x - 2y - 3z = 1$
$x - y + 7z = 2$

34. $2x + y = 1$
$3y + z = 2$
$2z + w = 3$
$5x + w = 4$

35. Specify the number of second order determinants in the matrix:

$$\begin{Vmatrix} 1 & 5 & 8 & 1 \\ 2 & 0 & 7 & 0 \\ 0 & 2 & 3 & 5 \end{Vmatrix}$$

36. Specify the number of third order determinants in the matrix of problem 35.

37. Specify the number of determinants of each order from one to six in a square matrix having six rows and six columns.

38. Specify the number of determinants of each order from one to five in a matrix having five rows and six columns.

39. Determine the rank of the matrix of problem 35.

40. Determine the rank of the determinant in problem 26.

41. Determine the rank of the determinant in problem 29.

42. Determine the rank of the following matrix:

$$\begin{Vmatrix} 2 & 1 & 3 & 7 & 9 & 11 \\ 4 & 1 & 2 & 1 & 1 & 2 \\ 6 & 2 & 7 & 1 & 5 & 9 \\ 6 & 2 & 5 & 8 & 10 & 13 \\ 10 & 3 & 9 & 2 & 6 & 11 \end{Vmatrix}$$

Discuss the following systems of equations:

43. $x + 2y + 3z = 5$
$2x + y - 7z = 2$
$3x + 3y - 4z = 1$

44. $5x - 2y + 3z = 2$
$3x + y + 4z = -1$
$4x - 3y + z = 3$

45. $x + 2y - 5z = 3$
$2x + y - 3z = 2$
$3x + 3y - 8z = 5$
$2x + 4y - 10z = 6$
$3y - 7z = 4$

46. $x + 2y + 2z - w = 3$
$2x - y - z + 7w = 4$
$3x + y + z + 6w = 6$

47. $x + 2y - 3z = 0$
$4x + 2y - 8z = 0$
$5x + 4y - 11z = 0$

48. $3x - 2y + 3z = 0$
$4x + 2y + z = 0$
$x + 2y - 8z = 0$

49. $2x - y + 3z + w = 0$
$4x + 3y - 8z + 2w = 0$
$6x + 2y - 5z + 3w = 0$
$8x + y - 2z + 4w = 0$

CHAPTER 3

DIMENSIONAL ANALYSIS

3.1 Introduction. Man's study of natural philosophy, or the laws of nature, has been based on two points of view. An investigation into why certain actions are found in nature and the causes underlying these actions is the heart of metaphysics. A study to determine how natural processes take place so that they can be prevented or brought into being at the will of the student is the aim of the present day scientist and makes up the field of science in its broadest sense.

In order to make a study of natural phenomena it is necessary to be able to describe them so that they can be recognized whenever they occur. In horticulture we describe a flower by giving the colors, number of petals, sepals, stamens, etc. Color, petals, etc., are the **dimensions** by means of which we describe the flower. Whether we count petals singly, or in pairs, is determined by the **unit** chosen.

3.2 Dimensional Formulas. A bar of iron has length, breadth, thickness, surface area, volume, mass, etc. If we use the foot for our unit of length we find the length to be p, the breadth q, the thickness r, the area s, the volume v, the mass m. If instead of the foot we use the inch for our unit of length we find the length of the bar to be $12p$, the breadth $12q$, the thickness $12r$, the area $144s$, the volume $1728v$, the mass however is still m.

The numerical values of some of the dimensions depend on the unit of length chosen, and some do not; in the example given above the mass is independent of the unit of length chosen. The volume, area, and thickness depend differently on the unit of length chosen; in fact the relations are:

Thickness depends on L,
Area depends on L^2,
Volume depends on L^3.

This dependence on the unit of length chosen is indicated by the dimensional formulas:

$$\text{Length} = [L],$$
$$\text{Area} = [L^2],$$
$$\text{Volume} = [L^3].$$

A velocity can be expressed in miles per hour, or feet per second, or any of several ways. The numerical value of the velocity depends on the unit of length and also on the unit of time. If we increase the size of the unit of length the numerical velocity is decreased: v feet per second is the same as $v/3$ yards per second. If we increase the size of the unit of time the numerical velocity is increased: v feet per second is the same as $60v$ feet per minute. The dimensional formula for velocity is $v = [LT^{-1}]$ where L has the same significance as in the formula length $= [L]$ and T^{-1} shows that the dependence of velocity on the unit of time is just the inverse of the dependence on the unit of length.

We find that mechanical dimensions, such as force, acceleration, weight, moment of inertia, etc., can be related to three basic dimensions. Length $[L]$, time $[T]$, and mass $[M]$ are usually used as the fundamental dimensions, and formulas which depend on mass, length, and time as fundamental dimensions are said to be based on the MLT system; see Table III–1.

TABLE III–1

Physical Quantity	*Dimensional Formula*	
	IRTL	MLTI
Length	L	L
Area	L^2	L^2
Volume	L^3	L^3
Time	T	T
Velocity	LT^{-1}	LT^{-1}
Acceleration	LT^{-2}	LT^{-2}
Plane angle	Numeric	Numeric
Angular velocity	T^{-1}	T^{-1}
Angular acceleration	T^{-2}	T^{-2}
Frequency	T^{-1}	T^{-1}
Mass	$I^2RT^3L^{-2}$	M
Force	I^2RTL^{-1}	MLT^{-2}
Impulse	$I^2RT^2L^{-1}$	MLT^{-1}
Momentum	$I^2RT^2L^{-1}$	MLT^{-1}
Torque	I^2RT	ML^2T^{-2}
Moment of inertia	I^2RT^3	ML^2
Moment of momentum	I^2RT^2	ML^2T^{-1}
Pressure	I^2RTL^{-3}	$ML^{-1}T^{-2}$
Pressure gradient	I^2RTL^{-4}	$ML^{-2}T^{-2}$
Energy	I^2RT	ML^2T^{-2}
Energy density	I^2RTL^{-3}	$ML^{-1}T^{-2}$
Power	I^2R	ML^2T^{-3}
Density	$I^2RT^3L^{-5}$	ML^{-3}
Viscosity	$I^2RT^2L^{-3}$	$ML^{-1}T^{-1}$
Gravitational constant	$I^{-2}R^{-1}T^{-5}L^5$	$M^{-1}L^3T^{-2}$

TABLE III-2

Physical Quantity	*Dimensional Formula*	
	IRTL	MLTI
Current	I	I
Current density	IL^{-2}	IL^{-2}
Resistance	R	$ML^2T^{-3}I^{-2}$
Resistivity	RL	$ML^3T^{-3}I^{-2}$
Conductivity	$R^{-1}L^{-1}$	$M^{-1}L^{-3}T^3I^2$
Conductance	R^{-1}	$M^{-1}L^{-2}T^3I^2$
Reactance	R	$ML^2T^{-3}I^{-2}$
Impedance	R	$ML^2T^{-3}I^{-2}$
Susceptance	R^{-1}	$M^{-1}L^{-2}T^3I^2$
Admittance	R^{-1}	$M^{-1}L^{-2}T^3I^2$
Permittivity	$R^{-1}TL^{-1}$	$M^{-1}L^{-3}T^4I^2$
Capacitance	$R^{-1}T$	$M^{-1}L^{-2}T^4I^2$
Elastance	RT^{-1}	$ML^2T^{-4}I^{-2}$
Elastivity	$RT^{-1}L$	$ML^3T^{-4}I^{-2}$
Permeability	RTL^{-1}	$MLT^{-2}I^{-2}$
Self inductance	RT	$ML^2T^{-2}I^{-2}$
Mutual inductance	RT	$ML^2T^{-2}I^{-2}$
Reluctance	$R^{-1}T^{-1}$	$M^{-1}L^{-2}T^2I^2$
Permeance	RT	$ML^2T^{-2}I^{-2}$
Reluctivity	$R^{-1}T^{-1}L$	$M^{-1}L^{-1}T^2I^2$
Coupling coefficient	Numeric	Numeric
Power factor	Numeric	Numeric
Electric charge	IT	IT
Charge density	ITL^{-3}	ITL^{-3}
Voltage	IR	$ML^2T^{-3}I^{-1}$
Voltage gradient	IRL^{-1}	$MLT^{-3}I^{-1}$
Magnetomotive force	I	I
Magnetic intensity	IL^{-1}	IL^{-1}
Magnetic flux	IRT	$ML^2T^{-2}I^{-1}$
Magnetic flux density	$IRTL^{-2}$	$MT^{-2}I^{-1}$

In thermodynamics it is usually desirable to introduce a fourth fundamental dimension, temperature θ; this gives the MLTθ system. In electric studies it is often customary to introduce permittivity ϵ, or permeability μ, as the fourth fundamental dimension giving the MLTϵ and MLTμ systems. There are two other systems useful in electric studies; one is based on mass, length, time, and current as fundamental dimensions giving the MLTI system; the other is based on current, resistance, time, and length as fundamental dimensions giving the IRTL system.

Tables III-1 and III-2 give dimensional formulas for some common physical quantities based both on the MLTI and the IRTL systems. Table III-1 contains mechanical quantities whereas Table III-2 includes electrical and magnetic quantities.

3.3 Change of System. The tables in this chapter give dimensional formulas in the MLTI and the IRTL systems and if a formula is required in another system it is necessary to know how to change from one system to another. If the dimensional formula of a given physical quantity can be written in two systems of dimensions the systems are said to be equivalent with respect to that physical quantity. Since energy $= [I^2RT] = [ML^2T^{-2}]$ can be written in the IRTL and the MLT systems, the IRTL and MLT systems are equivalent with respect to energy. Voltage $= [IR]$ in the IRTL systems but no formula is possible in the MLT system; therefore these systems are not equivalent with respect to voltage. It happens that any formula that can be written in the IRTL system can also be written in the MLTI system and vice versa; therefore these systems are said to be equivalent. If the formula for every physical quantity can be written in a particular system, that system is called a universal system. Obviously two universal systems are equivalent. The systems MLTIθ and IRTLθ, where θ is temperature, appear to be universal systems.

Let one set of fundamental dimensions be X_1, X_2, X_3 (we consider three fundamental dimensions to simplify the explanation; the process with two, four, or five would be the same). Let the dimensional formula for a given physical quantity Q be

$$Q = X_1^a X_2^b X_3^c. \tag{3.1}$$

It is required to obtain the dimensional formula for Q based on the system of fundamental dimensions Y_1, Y_2, Y_3.

$$Q = Y_1^r Y_2^s Y_3^t \tag{3.2}$$

where the dimensional formulas for the new fundamental dimensions based on the old basic dimensions are

$$\begin{aligned} Y_1 &= X_1^{a_{11}} X_2^{a_{12}} X_3^{a_{13}}, \\ Y_2 &= X_1^{a_{21}} X_2^{a_{22}} X_3^{a_{23}}, \\ Y_3 &= X_1^{a_{31}} X_2^{a_{32}} X_3^{a_{33}}. \end{aligned} \tag{3.3}$$

The procedure is to substitute equations (3.3) into equation (3.2).

$$\begin{aligned} Q &= (X_1^{a_{11}} X_2^{a_{12}} X_3^{a_{13}})^r (X_1^{a_{21}} X_2^{a_{22}} X_3^{a_{23}})^s (X_1^{a_{31}} X_2^{a_{32}} X_3^{a_{33}})^t \\ &= X_1^{a_{11}r + a_{21}s + a_{31}t} X_2^{a_{12}r + a_{22}s + a_{32}t} X_3^{a_{13}r + a_{23}s + a_{33}t} \end{aligned} \tag{3.4}$$

Comparing equation (3.4) with equation (3.1), we have

$$\begin{aligned} a_{11}r + a_{21}s + a_{31}t &= a, \\ a_{12}r + a_{22}s + a_{32}t &= b, \\ a_{13}r + a_{23}s + a_{33}t &= c. \end{aligned} \tag{3.5}$$

If

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = D \neq 0 \tag{3.6}$$

the systems are equivalent and we have

$$r = \frac{1}{D}\begin{vmatrix} a & a_{21} & a_{31} \\ b & a_{22} & a_{32} \\ c & a_{23} & a_{33} \end{vmatrix},$$

$$s = \frac{1}{D}\begin{vmatrix} a_{11} & a & a_{31} \\ a_{12} & b & a_{32} \\ a_{13} & c & a_{33} \end{vmatrix}, \tag{3.7}$$

$$t = \frac{1}{D}\begin{vmatrix} a_{11} & a_{21} & a \\ a_{12} & a_{22} & b \\ a_{13} & a_{23} & c \end{vmatrix}.$$

Example 1. From the dimensional formula for force in the MLTI system find the dimensional formula for force in the IRTL system. We have the formula for force given as

$$F = MLT^{-2} = M^1L^1T^{-2}I^0. \tag{3.8}$$

The formulas needed for the transformation are

$$\begin{aligned} I &= I &&= M^0L^0T^0I^1, \\ R &= ML^2T^{-3}I^{-2} &&= M^1L^2T^{-3}I^{-2}, \\ T &= T &&= M^0L^0T^1I^0, \\ L &= L &&= M^0L^1T^0I^0, \end{aligned} \tag{3.9}$$

We require p, q, r, s in

$$F = I^pR^qT^rL^s. \tag{3.10}$$

The exponents in equations (3.9) give

$$D = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -3 & 1 & 0 \\ 1 & -2 & 0 & 0 \end{vmatrix} = 1. \tag{3.11}$$

Note that the exponents in the first of equations (3.9) give the first column in determinant D; the exponents in the second equation give the second column, etc. To find p we replace the first column of D by the exponents in equation (3.8) and write

$$p = \frac{1}{D}\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ -2 & -3 & 1 & 0 \\ 0 & -2 & 0 & 0 \end{vmatrix} = 2 \tag{3.12}$$

$$q = \frac{1}{D}\begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1 \tag{3.13}$$

$$r = \frac{1}{D}\begin{vmatrix} 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & -3 & -2 & 0 \\ 1 & -2 & 0 & 0 \end{vmatrix} = 1 \tag{3.14}$$

$$s = \frac{1}{D}\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & -3 & 1 & -2 \\ 1 & -2 & 0 & 0 \end{vmatrix} = -1 \tag{3.15}$$

Therefore,

$$F = I^2RTL^{-1}.$$

Example 2. From the dimensional formula for force in the IRTL system find the dimensional formula for force in the MLTI system. This is the converse of the first example.

Given

$$F = I^2RTL^{-1}. \tag{3.17}$$

The transformation formulas are

$$\begin{aligned} M &= I^2RT^3L^{-2} &&= I^2R^1T^3L^{-2}, \\ L &= L &&= I^0R^0T^0L^1, \\ T &= T &&= I^0R^0T^1L^0, \\ I &= I &&= I^1R^0T^0L^0. \end{aligned} \tag{3.18}$$

Required p, q, r, s in

$$F = M^pL^qT^rI^s. \tag{3.19}$$

The exponents on the right of equations (3.18) give for D

$$D = \begin{vmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \end{vmatrix} = 1. \tag{3.20}$$

Substitute the exponents on the right of equation (3.17) for successive columns of D giving the required solutions

$$p = \frac{1}{D}\begin{vmatrix} 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \end{vmatrix} = 1, \tag{3.21}$$

$$q = \frac{1}{D}\begin{vmatrix} 2 & 2 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ -2 & -1 & 0 & 0 \end{vmatrix} = 1, \tag{3.22}$$

$$r = \frac{1}{D}\begin{vmatrix} 2 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 \end{vmatrix} = -2, \tag{3.23}$$

$$s = \frac{1}{D}\begin{vmatrix} 2 & 0 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 3 & 0 & 1 & 1 \\ -2 & 1 & 0 & -1 \end{vmatrix} = 0. \tag{3.24}$$

Therefore, the formula for force in the MLTI system is

$$F = MLT^{-2}. \tag{3.25}$$

3.4 Change of Units. Dimensional formulas are of assistance in computing the effect of a change in the units used for measuring a physical quantity. For example, to find the magnitude of a velocity in feet per second equal to 30 miles per hour, let v feet per second be the required velocity.

Then

$$v \text{ feet per second} = 30 \text{ miles per hour}$$

$$v[L_1T_1^{-1}] = 30[L_2T_2^{-1}]$$

where $[L_1T_1^{-1}]$ is unit velocity in feet per second and $[L_2T_2^{-1}]$ is unit velocity in miles per hour.

$$v = 30[L_2/L_1][T_2/T_1]^{-1}.$$

L_2/L_1 is one mile divided by one foot or 5280, and T_2/T_1 is one hour divided by one second or 3600.

Therefore,

$$v = 30\,\frac{5280}{3600} = 44.$$

The required velocity is 44 feet per second.

3.5 Checking Physical Formulas. We would not think of adding time in seconds to area in square feet, or a force in pounds to a current in amperes. There might be a temptation to add weight in pounds to mass in grams, or magnetomotive force to electromotive force. We find that dimensional formulas are of great assistance in telling us what is permissible. If two physical quantities are added they must have the same dimensions. Suppose a formula gives a physical quantity as depending on the surface of a cube plus the volume of the cube. Consider a one foot cube. The surface is six and the volume is one. The total is seven. The surface contributes six sevenths and the volume contributes one seventh. Now, instead of measuring the cube in feet, we measure it in inches and find the area 864 and the volume 1728. The surface now contributes one third of the total and the volume two thirds of the total. But a physical process is independent of the units chosen to measure it; therefore the formula is incorrect. This principle was first expressed by Fourier in 1822 and leads us to the applications of dimensional analysis which follow.

The formula for the distance covered by a falling body is $s = \frac{1}{2}gt^2$. According to this principle, s and gt^2 must have the same dimensional formula. Now $s = [L]$ and $gt^2 = [LT^{-2}][T^2] = [L]$ and the formula is dimensionally correct. It is apparent that this is a one-way check. There is no way to tell dimensionally whether the constant $\frac{1}{2}$ should be $\frac{1}{2}$ or $\frac{1}{4}$ or what. In other words if a formula does not check dimensionally it is incorrect; but it is true that if a formula does check dimensionally it may still be wrong. Since there is no way to distinguish dimensionally between torque and energy, exceptional cases may arise from time to time where a misleading conclusion is drawn. If dimensional analysis is used as it should be, as a guide and aid rather than as an infallible tool, there will be no trouble.

In expressions such as $\sin X$, $\cosh X$, $\log X$, etc., X must be dimensionless. This is evident if we consider the series expansion of one of the above, say $\sin X$.

$$\sin X = X - \frac{X^3}{3!} + \frac{X^5}{5!} \cdots.$$

According to Fourier's principle each term in the series has the same dimensional formula. In particular X and X^3 have the same dimensions. Suppose the dimensional formula for X is

$$X = M^a L^b T^c.$$

The formula for X^3 will be

$$X^3 = M^{3a} L^{3b} T^{3c}.$$

Fourier's principle tells us that X and X^3 have the same dimensions, and this means that $a = 3a$, $b = 3b$, and $c = 3c$. These equations are satisfied only if $a = b = c = 0$, and this means that X is dimensionless.

3.6 Dimensional Constants. If we try to check the formula for the force of attraction f between two masses m_1 and m_2 separated by a distance r, we have

$$f = \frac{m_1 m_2}{r^2}. \tag{3.25}$$

Using the MLT system (Table III-1)

$$\begin{aligned} f &= [MLT^{-2}], \\ m_1 &= [M], \\ m_2 &= [M], \\ r &= [L]. \end{aligned}$$

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Therefore, equation (3.25) is dimensionally

$$[MLT^{-2}] \text{ should equal } [M][M][L^2]^{-1} = [M^2L^{-2}]$$

and we do not have a check.

The formula written above (equation 3.25) is wrong; it should be

$$f = G\frac{m_1m_2}{r^2} \tag{3.26}$$

where G is the gravitational constant and has the dimensions $M^{-1}L^3T^{-2}$. There is no trouble checking equation (3.26) using the formula for the dimensional constant.

A similar situation arises in the formula for the force between two electric charges and for the force between two magnetic poles. We must introduce dimensional constants, ϵ the permittivity and μ the permeability, to make the respective formulas check. It is interesting to note that this condition occurs in the three cases of "action at a distance." After these three dimensional constants have been introduced and defined they can be treated just like any other physical quantity; in fact the constant μ, or permeability, is used as a fundamental dimension in the MLTμ system.

3.7 Derivation of Physical Formulas. Fourier's principle, which we use to check equations dimensionally, enables us to make important steps in the derivation of unknown formulas. If we know what physical quantities are involved, and specify that the formula relating them is dimensionally correct, we can sometimes obtain the desired formula correct except for a proportionality constant. The method in its simplest form is illustrated in the following example. The force of a lifting magnet depends on the area of the gap A, on the flux density B, and on the permeability μ. We write the force as a constant multiplied by the quantities A, B, and μ, each raised to an unknown power.

$$F = kA^aB^b\mu^c. \tag{3.27}$$

Dimensionally, this is (see Tables III-1 and III-2),

$$[I^2RTL^{-1}] = [L^2]^a[IRTL^{-2}]^b[RTL^{-1}]^c. \tag{3.28}$$

If this formula is correct dimensionally the exponents of I, R, T, and L on both sides of the equation must be the same. This gives four equations to solve for a, b, and c.

$$\begin{array}{ll} \text{From } I: & 2 = b. \\ \text{From } R: & 1 = b + c. \\ \text{From } T: & 1 = b + c. \\ \text{From } L: & -1 = 2a - 2b - c. \end{array} \tag{3.29}$$

These equations can be solved for a, b, and c, giving $a = 1, b = 2, c = -1$. The relation we seek is therefore

$$F = \frac{kAB^2}{\mu}.$$

We can construct a magnet and determine k by measuring F, A, B, and μ, thus giving a complete solution by means of a single test. Without the aid of dimensional analysis the procedure would be to construct magnets having different areas to find how the force depends on A; then to set up different flux densities in one magnet to determine how the force depends on B. The method of finding how the force depends on μ would be even more difficult. When we consider that dimensional analysis gives us a complete solution depending on one experimental measurement, it is obvious that there is a tremendous saving in time and effort. Attention is called to the fact that we can find the force of a lifting magnet without resorting to the theory of magnets other than to say: "maybe the formula contains the area, flux density, and permeability!"

3.8 Introduction to Buckingham's π Theorem. In the example of the preceding paragraph we obtained four equations to solve for three unknowns a, b, and c. If the universal systems of dimensions contain five fundamental dimensions we can never get more than five equations. There is no limit on the number of unknowns that may be required for more complicated problems. The procedure to follow in cases where the equations do not enable us to solve directly for the exponents is made clear if we work the above example as a special case of the general method. The only physical quantities involved are force, F; flux density, B; area, A; and permeability, μ. These must be related by some law which can be expressed as follows:

$$\Phi(F, B, A, \mu) = \text{constant} = \text{numeric}. \tag{3.30}$$

The right-hand side of the above equation is dimensionless; therefore the left-hand side must be a numeric also. In the unknown function $\Phi(F, B, A, \mu)$ we do know that the four variables are so related that each term is dimensionless. No term in $\Phi(F, B, A, \mu)$ can contain just one of the variables, say B, because if we transferred it to the right we would have B + numeric, the sum of two quantities not having the same dimensions. Then any term in $\Phi(F, B, A, \mu)$, containing one of the variables, contains enough of the others raised to the right powers to make the combination dimensionless. Or dimensional analysis tells us that

$$[F]^a[B]^b[A]^c[\mu]^d = \text{numeric}. \tag{3.31}$$

Substitution from the tables gives

$$[I^2RTL^{-1}]^a[IRTL^{-2}]^b[L^2]^c[RTL^{-1}]^d = I^0R^0T^0L^0. \tag{3.32}$$

Comparing exponents on both sides of the equation we find

$$\begin{aligned} &\text{From } I: & 2a + b &= 0. \\ &\text{From } R: & a + b + d &= 0. \\ &\text{From } T: & a + b + d &= 0. \\ &\text{From } L: & -a - 2b + 2c - d &= 0. \end{aligned} \tag{3.33}$$

We have four homogeneous equations in four unknowns. The matrix of the system is

$$\left\| \begin{matrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & -2 & 2 & -1 \end{matrix} \right\|. \tag{3.34}$$

This is of rank three; therefore we have three independent equations. Since the determinant obtained by crossing out the first column and second row of the matrix in equation (3.34) is not zero, we can solve for b, c, and d in terms of a. Doing this we find $b = -2a$, $c = -a$, $d = a$. Substituting on the left side of equation (3.31), we have

$$[F]^a[B]^b[A]^c[\mu]^d = \left[\frac{F\mu}{B^2A}\right]^a = \text{numeric}. \tag{3.35}$$

This means that if F appears in a term of $\Phi(F, B, A, \mu) = \text{constant}$, it will be multiplied by μ and divided by B^2A. If F^2 appears it will be multiplied by μ^2 and divided by B^4A^2. The quantity $(F\mu B^{-2}A^{-1})^a$ is dimensionless for any value of a. Let $F\mu B^{-2}A^{-1} = x$; then in place of the equation $\Phi(F, B, A, \mu) = \text{constant}$, where Φ is a function of four variables, we can write $f_1(x^a) = \text{constant}$ or more simply $f(x) = \text{constant}$. But $f(x) = \text{constant}$ means $x = \text{constant}$. Since $x = F\mu B^{-2}A^{-1}$, we have $F\mu B^{-2}A^{-1} = \text{constant} = k$ or

$$F = \frac{kB^2A}{\mu}.$$

Example 1. Find by dimensional analysis the relation between the velocity of sound in a gas and the pressure and density of the gas.

$$\Phi(v, p, \rho) = \text{constant}. \tag{3.36}$$

$$[v]^a[p]^b[\rho]^c = \text{numeric}. \tag{3.37}$$

$$[LT^{-1}]^a[ML^{-1}T^{-2}]^b[ML^{-3}]^c = M^0L^0T^0. \tag{3.38}$$

$$\begin{aligned} &\text{From } M \text{ we have:} & b + c &= 0. \\ &\text{From } L \text{ we have:} & a - b - 3c &= 0. \\ &\text{From } T \text{ we have:} & -a - 2b &= 0. \end{aligned} \tag{3.39}$$

We have three homogeneous equations in three unknowns: a, b, and c. The matrix of the system is

$$\left\| \begin{matrix} 0 & 1 & 1 \\ 1 & -1 & -3 \\ -1 & -2 & 0 \end{matrix} \right\|. \tag{3.40}$$

This is of rank two. Since the determinant obtained by crossing out the third column and the third row is not zero we can solve for a and b in terms of c and obtain $a = 2c$, $b = -c$; therefore, in place of equation (3.36), we have

$$f_1[(v^2\rho p^{-1})^c] = \text{constant} \tag{3.41}$$

or

$$f(v^2\rho p^{-1}) = \text{constant} \tag{3.42}$$

which means $v^2\rho p^{-1} = \text{constant}$ or $v = k\sqrt{p\rho^{-1}}$.

In each of the examples above the final result is obtained as though after solving for the unknown exponents in terms of one exponent we set that exponent equal to unity.

Example 2. Let us consider a more involved problem, that of finding the air resistance of an airplane wing. It is known that the resistance F depends on the shape, size L, speed V, density of air ρ, and viscosity μ of air. We write the unknown relation as above

$$\Phi(F, L, V, \rho, \mu) = \text{constant}, \tag{3.43}$$

$$[F^a][L^b][V^c][\rho^d][\mu^e] = \text{numeric}, \tag{3.44}$$

$$[MLT^{-2}]^a[L]^b[LT^{-1}]^c[ML^{-3}]^d[ML^{-1}T^{-1}]^e = M^0L^0T^0. \tag{3.45}$$

$$\begin{aligned} &\text{From } M \text{ we have:} & a + d + e &= 0. \\ &\text{From } L \text{ we have:} & a + b + c - 3d - e &= 0. \\ &\text{From } T \text{ we have:} & -2a - c - e &= 0. \end{aligned} \tag{3.46}$$

We now have three homogeneous equations in five unknowns a, b, c, d, and e. The matrix of the system is

$$\left\| \begin{matrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -3 & -1 \\ -2 & 0 & -1 & 0 & -1 \end{matrix} \right\|. \tag{3.47}$$

This is of rank three. Since the determinant obtained by crossing out the first and fifth columns is not zero we can solve for b, c, and d in terms of a and e.

$$d = -a - e, \qquad c = -2a - e, \qquad b = -2a - e$$

and we can now substitute in the left side of equation (3.44) to obtain

$$F^aL^bV^c\rho^d\mu^e = F^aL^{-2a-e}V^{-2a-e}\rho^{-a-e}\mu^e \tag{3.48}$$

$$= \left(\frac{F}{L^2V^2\rho}\right)^a\left(\frac{\mu}{LV\rho}\right)^e. \tag{3.49}$$

In place of $\Phi(F, L, V, \rho, \mu) = \text{constant}$, we have

$$f_1\left[\left(\frac{F}{L^2V^2\rho}\right)^a\left(\frac{\mu}{LV\rho}\right)^e\right] = \text{constant} \tag{3.50}$$

where a and e are arbitrary. This can be written $f_1(x^a y^e) = \text{constant}$ if we make the following substitution,

$$x = \frac{F}{L^2V^2\rho}, \qquad y = \frac{\mu}{LV\rho}.$$

But $f_1(x^a y^e) = \text{constant}$ can be written more simply as $f(x, y) = \text{constant}$. That is, by means of dimensional analysis, we have changed the problem from one of finding an unknown function Φ of five variables into a problem of finding an unknown function of two variables.

In order to illustrate how $f_1(x^a y^e)$ is the same as $f(x, y)$, where a and e are arbitrary, consider the following form: let $f_1(x^a y^e) = (x^a y^e)^2 + \sin\,(x^a y^e) + \cos\,(x^a y^e)$; if in the first term $a = 1$, $e = 2$, in the second term $a = 0$, $e = 3$, and in the third term $a = 1$, $e = 0$, then $f_1(x^a y^e) = x^2y^4 + \sin y^3 + \cos x$. There would be no hesitation in saying $x^2y^4 + \sin y^3 + \cos x$ is a function of the two variables x and y, in other words $f(x, y)$. Therefore, we write

$$f\left(\frac{F}{L^2V^2\rho}, \frac{\mu}{LV\rho}\right) = \text{constant}. \tag{3.51}$$

Just as $f(x, y) = \text{constant}$ can be solved for x in terms of y giving $x = f_2(y)$, equation (3.51) can be solved for $FL^{-2}V^{-2}\rho^{-1}$ in terms of $\mu L^{-1}V^{-1}\rho^{-1}$ giving

$$\frac{F}{L^2V^2\rho} = f_2\left(\frac{\mu}{LV\rho}\right). \tag{3.52}$$

And finally we can solve for F and have

$$F = L^2V^2\rho f_2\left(\frac{\mu}{LV\rho}\right). \tag{3.53}$$

The form of the function f_2 depends on the shape of the wing and is probably rather complicated. If we have a model with the same shape as the actual wing and plan to test it in air, the equation for the model will be

$$F' = L'^2V'^2\rho f_2\left(\frac{\mu}{L'V'\rho}\right). \tag{3.54}$$

If this equation is divided into the equation above we have

$$\frac{F}{F'} = \frac{L^2V^2}{L'^2V'^2}\,\frac{f_2\left(\dfrac{\mu}{LV\rho}\right)}{f_2\left(\dfrac{\mu}{L'V'\rho}\right)}. \tag{3.55}$$

If we make $L'V'$ for the model equal to LV for the actual wing, then the unknown function will cancel out and equation (3.55) becomes

$$\frac{F}{F'} = \frac{L^2V^2}{L'^2V'^2} \tag{3.56}$$

and since we have made $L'V'$ equal to LV this turns out to be equal to one; therefore $F = F'$. If the model wing is one tenth the size of the actual wing we must test it at ten times the actual expected air speed and the resulting measured force will be the same size as the actual force experienced by the actual

wing. This result is not very useful for high speed planes; therefore the model is tested in a high pressure wind tunnel or an approximation to the unknown function in equation (3.54) must be made.

Example 3. As another example to illustrate this more general problem consider a long, direct current transmission line. The generator current I_G is known to depend on the load current I_L, the load voltage E_L, the resistance of the wire per unit length r, the leakage conductance per unit length g, and on the length of the line s. These quantities are related by an equation

$$\Phi(I_G, I_L, E_L, r, g, s) = \text{constant}. \tag{3.57}$$

We write

$$[I_G]^a[I_L]^b[E_L]^c[r]^d[g]^e[s]^f = \text{numeric}. \tag{3.58}$$

$$[I]^a[I]^b[IR]^c[RL^{-1}]^d[R^{-1}L^{-1}]^e[L]^f = I^0R^0L^0. \tag{3.59}$$

$$\begin{aligned} &\text{From } I \text{ we have:} && a + b + c = 0. \\ &\text{From } R \text{ we have:} && c + d - e = 0. \\ &\text{From } L \text{ we have:} && -d - e + f = 0. \end{aligned} \tag{3.60}$$

We have three homogeneous equations in six unknowns. The matrix of the system is

$$\left\| \begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{matrix} \right\|. \tag{3.61}$$

This is of rank three. Since the third order determinant obtained by crossing out the second, third, and fifth columns is not zero, we can solve for a, d, and e, in terms of b, c, and f.

$$a = -b - c, \qquad d = \frac{f - c}{2}, \qquad e = \frac{c + f}{2}. \tag{3.62}$$

$$I_G^aI_L^bE_L^cr^dg^es^f = \left(\frac{I_L}{I_G}\right)^b\left(\frac{E_L}{I_G}\sqrt{\frac{g}{r}}\right)^c(\sqrt{rg}\,s)^f. \tag{3.63}$$

Using the following substitutions

$$x = \frac{I_L}{I_G}, \qquad y = \frac{E_L}{I_G}\sqrt{\frac{g}{r}}, \qquad z = \sqrt{rg}\,s. \tag{3.64}$$

we can say that in place of the equation $\Phi(I_G, I_L, E_L, r, g, s) = \text{constant}$, we now have $f_1(x^by^cz^f) = \text{constant}$, where b, c, and f are arbitrary constants. This of course is the same thing as saying $f(x, y, z) = \text{constant}$. In place of an equation involving six variables we have an equation involving three variables. Our final solution is written

$$f\left(\frac{I_L}{I_G}, \frac{E_L}{I_G}\sqrt{\frac{g}{r}}, \sqrt{rg}\,s\right) = \text{constant}. \tag{3.65}$$

The actual form of the unknown function f can be found by other means to be

$$\left(\frac{I_L}{I_G}\right)\cosh\sqrt{rg}\,s + \left(\frac{E_L}{I_G}\sqrt{\frac{g}{r}}\right)\sinh\sqrt{rg}\,s = 1. \tag{3.66}$$

3.9 Buckingham's π Theorem. Some of the steps in the preceding examples can be left out to speed up the solution. The following set of directions will be found to apply to each of the above problems, and is

known as Buckingham's π theorem. To find a relation among certain physical quantities $X_1, X_2, \cdots X_n$ replace each of the physical quantities by its dimensional formula in the equation

$$[X_1]^a[X_2]^b[X_3]^c \cdots [X_n]^n = 1, \tag{3.67}$$

perform the indicated multiplication on the left, and equate the exponent of each dimension to zero. This gives r independent equations in the unknowns $a, b, c, \cdots n$ (r is equal to 1, 2, 3, 4, or 5 if the universal system has only five fundamental dimensions). Solve for r of the unknowns in terms of the remaining $n - r$ unknowns. Let the first of the $n - r$ unknowns equal one (or any number not zero) and the others equal zero, giving a set of values for $a, b, c, \cdots n$, which we call $a_1, b_1, c_1, \cdots n_1$. Then equate the second of the $n - r$ unknowns to one and the others including the first to zero; this gives $a_2, b_2, c_2, \cdots n_2$. Repeat this process for each of the $n - r$ unknowns giving $n - r$ sets of values for $a, b, c, \cdots n$.

Let

$$\begin{aligned} \pi_1 &= X_1^{a_1}X_2^{b_1}X_3^{c_1} \cdots X_n^{n_1} \\ \pi_2 &= X_1^{a_2}X_2^{b_2}X_3^{c_2} \cdots X_n^{n_2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \pi_{n-r} &= X_1^{a_{n-r}}X_2^{b_{n-r}}X_3^{c_{n-r}} \cdots X_n^{n_{n-r}}. \end{aligned} \tag{3.68}$$

The desired equation is now

$$f(\pi_1, \pi_2, \pi_3, \cdots \pi_n) = \text{constant}. \tag{3.69}$$

This equation can be solved for any of the variables, giving a new equation

$$\pi_1 = f_2(\pi_2, \pi_3, \cdots \pi_n). \tag{3.70}$$

To illustrate the application of the π theorem consider example 2 above: Having solved equations (3.46) for b, c, and d in terms of a and e we have

$$\begin{aligned} a &= a \\ b &= -2a - e \\ c &= -2a - e \\ d &= -a - e \\ e &= e \end{aligned} \tag{3.71}$$

First let $a = a_1 = 1$ and $e = e_1 = 0$, then let $a = a_2 = 0$ and $e = e_2 = 1$; this gives us

$$\begin{aligned} a_1 &= 1 & a_2 &= 0 \\ b_1 &= -2 & b_2 &= -1 \\ c_1 &= -2 & c_2 &= -1 \\ d_1 &= -1 & d_2 &= -1 \\ e_1 &= 0 & e_2 &= 1 \end{aligned} \tag{3.72}$$

This gives the following dimensionless products

$$\pi_1 = \frac{F}{L^2V^2\rho}, \qquad \pi_2 = \frac{\mu}{LV\rho}. \tag{3.73}$$

Equation (3.69) gives

$$f\left(\frac{F}{L^2V^2\rho}, \frac{\mu}{LV\rho}\right) = \text{constant} \tag{3.74}$$

which is the same as equation (3.51), and equation (3.70) gives

$$\frac{F}{L^2V^2\rho} = f_2\left(\frac{\mu}{LV\rho}\right) \tag{3.75}$$

which is the same as equation (3.53).

In using the π theorem we obtain a set of homogeneous equations from which we find values for the exponents of the physical quantities. At first it may be confusing to decide which exponents to solve for in terms of the others. Of the many selections we might make, some are impossible of solution and, among the possible ones, some will greatly simplify the problem.

An inspection of the matrix of the system tells which unknowns can be solved for in terms of the others. The rank of the matrix of the coefficients of the unknowns to be solved for must equal the rank of the matrix of the system. For example, we cannot solve for a, b, and c in terms of d, e, and f in example 3 because the matrix of the coefficients of a, b, and c is

$$\begin{Vmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{Vmatrix} \tag{3.76}$$

which is of rank two, while the matrix of the system for equations (3.60) is of rank three.

Sometimes it is better to solve for one unknown rather than another. In example 2 we wanted an expression for the force F on an airplane wing. If we had solved for a, b, and c in terms of d and e we would have

$$\begin{array}{lll} a = -d - e & a_1 = -1 & a_2 = -1 \\ b = 2d + e & b_1 = 2 & b_2 = 1 \\ c = 2d + e & c_1 = 2 & c_2 = 1 \\ d = d & d_1 = 1 & d_2 = 0 \\ e = e & e_1 = 0 & e_2 = 1 \end{array}$$

The dimensionless products would now be

$$\pi_1 = \frac{L^2V^2\rho}{F} \qquad \pi_2 = \frac{LV\mu}{F} \tag{3.78}$$

which would give

$$f\left(\frac{L^2V^2\rho}{F}, \frac{LV\mu}{F}\right) = \text{constant.} \tag{3.79}$$

And, finally,

$$\frac{L^2V^2\rho}{F} = f_2\left(\frac{LV\mu}{F}\right). \tag{3.80}$$

This can be written

$$F = \frac{L^2V^2\rho}{f_2\left(\dfrac{LV\mu}{F}\right)} \tag{3.81}$$

which is not much good since F appears in the functional expression in the denominator.

Since we want an expression for the force F, we want F to appear only once in the result so that a formal solution is possible. To make sure of this we do not solve for a, the exponent of F, in terms of the other unknowns. We are sure of a satisfactory solution if we solve for any three of b, c, d, and e in terms of the remaining one and a.

The examples given above should be worked out following the directions just given to make sure that the theorem is understood. A fourth example is now given to show how dimensionless quantities are handled.

Example 4. The time T of the swing of a pendulum is known to depend on its length L, mass M, and weight W, and the angular amplitude of the swing θ.

$$[T]^a[L]^b[M]^c[W]^d[\theta]^e = 1. \tag{3.82}$$

$$[T]^a[L]^b[M]^c[MLT^{-2}]^d = M^0L^0T^0. \tag{3.83}$$

Note θ is a numeric and therefore does not appear above.

$$\begin{aligned} &\text{From } T \text{ we have:} && a - 2d = 0. \\ &\text{From } L \text{ we have:} && b + d = 0. \\ &\text{From } M \text{ we have:} && c + d = 0. \end{aligned} \tag{3.84}$$

These are three homogeneous equations with five unknowns. The matrix of the system is

$$\begin{Vmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{Vmatrix}. \tag{3.85}$$

This is of rank three. Since the determinant obtained by crossing out the first and fifth columns is not zero we can solve for b, c, and d in terms of a and e.

This gives us

$$
\begin{array}{lll}
a = a & a_1 = 1 & a_2 = 0 \\
b = \dfrac{-a}{2} & b_1 = -\frac{1}{2} & b_2 = 0 \\
c = \dfrac{-a}{2} & c_1 = -\frac{1}{2} & c_2 = 0 \\
d = \dfrac{a}{2} & d_1 = \frac{1}{2} & d_2 = 0 \\
e = e & e_1 = 0 & e_2 = 1
\end{array}
$$

$$\pi_1 = T\sqrt{\frac{W}{ML}}, \qquad \pi_2 = \theta. \tag{3.86}$$

The resulting equation is

$$f\left(T\sqrt{\frac{W}{ML}}, \theta\right) = \text{constant} \tag{3.87}$$

or solving for π_1

$$T\sqrt{\frac{W}{ML}} = f_2(\theta) \tag{3.88}$$

which can be written

$$T = \sqrt{\frac{LM}{W}}\, f_2(\theta). \tag{3.89}$$

This is as far as dimensional analysis will take us. A few more comments on this problem will be of interest. Since $W = Mg$, this can be written

$$T = \sqrt{\frac{L}{g}}\, f_2(\theta). \tag{3.90}$$

For small values of $\theta, f_2(\theta)$ is practically constant, and we have then, if θ is small,

$$T = k\sqrt{\frac{L}{g}}, \tag{3.91}$$

an equation that is familiar to physics students.

3.10 Model Study. Let π_1, π_2, etc., be the π's for the actual process, and π_1', π_2', etc., be the corresponding relations for the model. Then the conditions that must hold are

$$\pi_1 = \pi_1', \quad \pi_2 = \pi_2', \quad \cdots \quad \pi_{n-r} = \pi_{n-r}'. \tag{3.92}$$

The quantity being determined appears in only one of these equations and that will be the **equation of the model.** The remaining $n - r - 1$ equations are the restrictions which are placed on the model to make the equation of the model true.

To illustrate this consider again the example of the airplane wing. We found, see equations (3.73),

$$\pi_1 = \frac{F}{L^2V^2\rho} \qquad \pi_2 = \frac{\mu}{LV\rho}. \tag{3.93}$$

For the model we have

$$\pi_1' = \frac{F'}{L'^2V'^2\rho'} \qquad \pi_2' = \frac{\mu'}{L'V'\rho'}. \tag{3.94}$$

If we make $\pi_2' = \pi_2$, that is,

$$\frac{\mu'}{L'V'\rho'} = \frac{\mu}{LV\rho} \tag{3.95}$$

then we will have $\pi_1' = \pi_1$, or

$$\frac{F'}{L'^2V'^2\rho'} = \frac{F}{L^2V^2\rho}. \tag{3.96}$$

If we test the model in air at atmospheric pressure $\mu' = \mu$ and $\rho' = \rho$ and equation (3.95) becomes

$$L'V' = LV. \tag{3.97}$$

Now, substituting $L'V' = LV$ and $\rho' = \rho$ in equation (3.96), we have

$$F' = F \tag{3.98}$$

which checks with equation (3.56).

PROBLEMS ON CHAPTER 3

1. Determine the dimensional formulas for additional physical quantities, for example, Young's modulus, Poisson's ratio, coefficient of friction.

2. Check the formulas in Table III–1 given in the MLT system by substituting for M, in the particular dimensional formula, the value of M in the IRTL system.

3. A series circuit containing inductance L and capacitance C is set in oscillation. Find how the period of oscillation depends on L and C, following the procedure of the first example in the text.

4. The value of the velocity of sound waves in a solid depends on Young's modulus E and on the density ρ. Find how these quantities enter into the formula for the velocity following the procedure in the first example of the text.

5. Solve the example on p. 50, using the general form of the π theorem.

6. Solve example 1, p. 52, in the text using the general form of the π theorem.

7. Solve example 3, p. 55, in the text using the general form of the π theorem.

8. Solve problem 3, using the general form of the π theorem.

9. Solve problem 4 using the general form of the π theorem.

10. The distance s passed through by a falling body is known to depend on the acceleration of gravity g, on the time of fall t, and on the initial velocity v at $t = 0$. Find how these quantities are related.

11. The energy stored in the field of an electromagnet is known to depend on the inductance L and on the current I. Find how these quantities are related.

12. The frequency of the sounds produced by a violin string depends on the force stretching the string T, on the length L, and on the density per unit length λ. Find how these quantities are related.

13. Find how the velocity of the water waves in a canal depends on the depth of the canal h, on the water density ρ, and on the acceleration of gravity g.

14. The rate at which electromagnetic energy passes through space in watts per square centimeter depends on the voltage gradient G, on the magnetic gradient H, and on the electromagnetic wave velocity c. Find how these quantities are related.

15. Determine how the energy stored in a condenser depends on the permittivity ϵ of the dielectric, on the area A of the plates, the distance between the plates h, and the applied voltage E. If it is known that the energy varies directly with the area of the plates what does the relation become?

16. The pressure gradient G which results when fluid flows in a smooth straight pipe depends on the diameter of the pipe D, on the velocity V, density ρ, and viscosity μ of the liquid. Find how these quantities are related.

17. The resistance force, F, of the water to the motion of a ship depends on the density ρ of the water, the area S of the wet surface, the length L, the velocity of the ship V, and the viscosity μ of the water. Find how these quantities are related.

18. In the preceding problem discuss the requirements if the dimensional results are to be used in the study of a model.

19. The thrust F of a propeller depends on the diameter D, the rate of revolution n, the speed of advance S, the density ρ and the viscosity μ of the water, and the acceleration of gravity g. Determine how these quantities are related.

20. Analyze the preceding problem on the basis of model study.

21. If in the preceding problem turbulent flow is assumed so that the thrust is independent of the viscosity how is the problem affected?

CHAPTER 4

COMPLEX NUMBERS AND HYPERBOLIC FUNCTIONS

4.1 Introduction. The terms complex number and imaginary number are rather forbidding names for certain kinds of numbers. These numbers are not imaginary, and when they are used they generally simplify a problem which would be quite complex otherwise. One of the fields in which complex numbers are used very extensively is alternating current circuits and machinery.

The positive integers may be considered as natural numbers. The idea of counting one's possessions leads to the use of positive integers. Addition and multiplication of positive integers result in positive integers, e.g., $4 + 5 = 9$, $5 \times 2 = 10$. Subtraction causes trouble if we try to subtract a large number from a smaller one. In order to make such a problem possible of solution we extend our number system to include negative integers. Negative integers and zero are invented to make subtraction always possible.

The process of division is sometimes possible in the number system consisting of positive and negative integers and zero, e.g., $42/6 = 7$, but $21/4$ is impossible of solution unless we extend our number system to include rational numbers. The invention of rational numbers makes it possible to divide one number by another with the one exception that division by zero is not defined.

Evolution, or the extraction of roots, e.g., square root, is not always possible in the system of rational numbers. The diagonal of a unit square is a definite length but it cannot be expressed as a rational number. We therefore extend the number system by inventing irrational numbers to make the extraction of roots of positive numbers possible.

In the system of positive and negative, rational and irrational numbers and zero, it is impossible to extract the square root of a negative number. We therefore extend the system of numbers to include imaginary and complex numbers. This last extension of the number system is really no more intangible than the extensions described above.

***4.2** ***i*-Numbers.** If we consider the integers marked along an axis as in Fig. 4–1, we note that when any integer is multiplied by -1 the result is on the other side of, and at the same distance from, zero, e.g., $(-1)(3) = -3$, $(-1)(-5) = 5$, etc. Then multiplication by -1 can be considered

as the same as a rotation of 180° about zero in a counterclockwise direction. Let us call the axis in Fig. 4-1 the R axis and include an i axis as in Fig. 4-2.

Consideration of Fig. 4-2 shows that we can obtain $i2$ by rotating 2 counterclockwise through 90° about zero as a pivot. i is obtained by rotating 1 about zero through an angle of 90° counterclockwise. In general ik is obtained by rotating k counterclockwise through an angle of 90°

FIG. 4-1

about zero as a pivot. We say that multiplication by i produces a rotation of 90° just as multiplication by -1 produces a rotation of 180°. If we multiply by i twice, the rotation will be through twice 90° or 180°, then multiplying by i^2 is the same as multiplying by -1. Applying these to $+1$ we have $i^2 = -1$, or $i = \sqrt{-1}$.

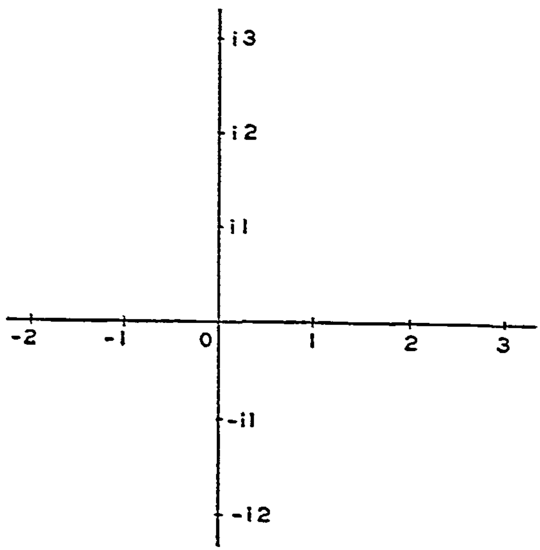

FIG. 4-2

*4.3 **The Operator i.** If multiplication by i produces a rotation of 90°, then i^2 produces a rotation of 180°, i^3 a rotation of 270° or $-90°$, and i^4 a rotation of 360°. This gives the following set of relations:

$$\begin{array}{ll} i^2 = -1 & i^3 = -i \\ i^4 = 1 & i^5 = i \\ i^6 = -1 & i^7 = -i \\ i^8 = 1 & \text{etc.} \end{array} \tag{4.1}$$

If we multiply a number by $i2$ the result is to rotate the number through 90° and move to one twice as far from the origin, e.g., $(i2)(4) = i8$, $(i2)(-3) = -i6$, etc. In general, multiplication by ik rotates through 90° counterclockwise and moves k times as far from zero.

Let z be an operator such that multiplication by z rotates through 45°, then multiplication by z twice will produce a rotation of 90°. Multiplying $+1$ by z^2 is the same as multiplying $+1$ by i; this gives $z^2 = i$, or $z = \sqrt{i}$. This gives a point on the 45°-line between the R axis and the i axis in Fig. 4–2. This point is called a complex number. Any point in the plane of Fig. 4–2 is a complex number; the points on the R axis are called real numbers, and the points on the i axis are called i-numbers or imaginary numbers. Both real numbers and imaginary numbers are complex numbers.

It is common in electrical engineering to reserve the letter i to indicate electric current and electrical engineers therefore use the letter j where we use the letter i. Thus imaginary numbers are sometimes called J-numbers, and $j = \sqrt{-1}$.

4.4 Complex Numbers. Having defined complex numbers as points in the plane of Fig. 4–2, the question of how to designate a particular number should be answered. In Fig. 4–3 the point P locates a complex number. We can fix the point P by noting that it is a distance A to the right of the i axis and a distance B above the R axis. We write this $A + iB$ or $2 + i3$, etc. A or 2 indicates that the point is a distance A or 2 units from the i axis and B or 3 indicates that the point is B or 3 units above the R axis. If C is the length of the line from 0 to P, and the line makes an angle θ with the R axis, then $A + iB = C\cos\theta + iC\sin\theta = C(\cos\theta + i\sin\theta)$. $A + iB$ is called the orthogonal or Cartesian form of the complex number. $C(\cos\theta + i\sin\theta)$ is the trigonometric form of the complex number. $\cos\theta + i\sin\theta$ is often abbreviated $\operatorname{cis}\theta$; therefore, $C\operatorname{cis}\theta$ is also the trigonometric form. $C\operatorname{cis}\theta$ is sometimes indicated as $C\underline{/\theta}$ which may be called the polar form or circular form of the complex number. In all these forms C is known as the modulus or absolute value and θ is the argument or angle.

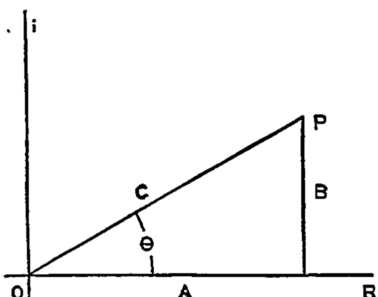

FIG. 4–3

***4.5 Algebra of Complex Numbers.** The sum of two complex numbers $a + ib$ and $c + id$ is found by adding the real parts to give the real part of the sum, and adding the imaginary parts to give the imaginary part of

the sum, i.e., $(a + ib) + (c + ib) = (a + c) + i(b + d)$, $(2 + i3) + (5 + i9) = 7 + i12$. The difference of two complex numbers is obtained by taking the difference of the real parts for the real part of the result and the difference of the imaginary parts for the imaginary part of the result.

$$(a + ib) - (c + id) = (a - c) + i(b - d),$$
$$(2 + i3) - (1 - i2) = (2 - 1) + i(3 + 2) = 1 + i5.$$

The multiplication and division of two complex numbers follow the rules of ordinary algebra with the additional relation $i^2 = -1$. Normally multiplication and division are performed with the numbers written in terms of r and θ; this is described in the next section. These processes can be performed in orthogonal form as follows:

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc),$$
$$(4 + i3)(2 - i3) = 17 - i6.$$

For division, we have

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)} = \frac{ac + bd}{c^2 + d^2} + i\frac{(bc - ad)}{(c^2 + d^2)}. \tag{4.2}$$

A numerical example follows

$$\frac{17 - i6}{4 + i3} = 2 - i3.$$

4.6 Exponential Form. There is another way of writing complex numbers that is very useful in analyzing problems in complex numbers. MacLaurin's series for the exponential term e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \cdots. \tag{4.3}$$

If we let $x = i\theta$, then $x^2 = -\theta^2$, $x^3 = -i\theta^3$, $x^4 = \theta^4$, etc. This substitution in equation (4.3) gives

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} \cdots. \tag{4.4}$$

Now MacLaurin's series for $\sin\theta$ and $\cos\theta$ are:

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots, \tag{4.5}$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} \cdots. \tag{4.6}$$

If equation (4.5) is multiplied throughout by i and then added to equation

(4.6), the result on the right will be found equal to the right side of equation (4.4); therefore, we have the important relation

$$\cos\theta + i\sin\theta = e^{i\theta}. \tag{4.7}$$

This is known as Euler's theorem.

As a résumé we have the following ways of writing a complex number:

$$a + ib = r(\cos\theta + i\sin\theta) = r \operatorname{cis}\theta = r\underline{/\theta} = r\,e^{i\theta}$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\theta = \arctan b/a$.

Multiplication and division are quite simple when the numbers are written in the exponential form $(A\,e^{i\theta})(B\,e^{i\phi}) = AB\,e^{i(\theta+\phi)}$; and $(A\,e^{i\theta})/(B\,e^{i\phi}) = (A/B)\,e^{i(\theta-\phi)}$.

Since $i = 0 + i = \cos\pi/2 + i\sin\pi/2 = e^{i(\pi/2)}$, multiplication by i has the effect of rotation through $\pi/2$ radians or 90° counterclockwise. This checks with the early discussion and therefore our algebra is consistent.

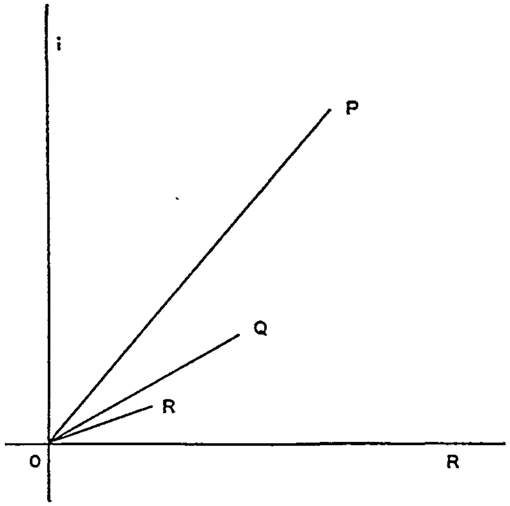

FIG. 4–4

4.7 Complex Numbers as Operators. In Fig. 4–4 we have plotted the complex numbers $Q = 4\operatorname{cis}30°$, $R = 2\operatorname{cis}20°$, and their product $P = 8\operatorname{cis}50°$. It is sometimes convenient to consider the process of multiplication as a rotation and magnification. When we multiply $Q = 4\operatorname{cis}30°$ by $R = 2\operatorname{cis}20°$ we rotate Q through an angle of 20° counterclockwise and double its length. From this point of view R is an operator which transforms Q into P in Fig. 4–4. In like manner $Q = 4\operatorname{cis}30°$ can be considered as the operator which rotates $R = 2\operatorname{cis}20°$ through 30° counter-

clockwise and increases its length to four times what it was, giving $P = 8 \text{ cis } 50°$ as the result.

In the case of division the operation is the inverse of multiplication. If the complex number $P = 8 \text{ cis } 50°$ is divided by the operator $R = 2 \text{ cis } 20°$, this rotates P through an angle of 20° clockwise and reduces its length to one half what it was. Therefore, division by $2 \text{ cis } 20°$ is the same as multiplication by $0.5 \text{ cis } -20°$.

4.8 Powers of Complex Numbers. The problem of raising a complex number to an integer power presents no difficulty when the number is written in exponential form.

$$(A\,e^{i\theta})^n = A^n\,e^{in\theta}. \tag{4.8}$$

Therefore $[A(\cos\theta + i\sin\theta)]^n = A^n(\cos n\theta + i\sin n\theta)$. The formula

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \tag{4.9}$$

is known as De Moivre's theorem. A numerical example of a complex number raised to an integer power is

$$(5 \text{ cis } 10°)^3 = 125 \text{ cis } 30°.$$

4.9 Roots of Complex Numbers. To extract the nth root of $A\,e^{i\theta}$ involves the solution of the equation $x^n - A\,e^{i\theta} = 0$. This is an equation of the nth degree and has n distinct roots. The problem is readily solved if we note that

$$A\,e^{i\theta} = A\,e^{i(\theta+2\pi)} = A\,e^{i(\theta+4\pi)} = \text{etc.} \tag{4.10}$$

We do the inverse of what we did in the preceding section on each of these expressions and obtain

$$A^{\frac{1}{n}}\,e^{\frac{\theta}{n}},\quad A^{\frac{1}{n}}\,e^{\frac{i(\theta+2\pi)}{n}},\quad \cdots\quad A^{\frac{1}{n}}\,e^{i\frac{\theta+2n\pi}{n}}. \tag{4.11}$$

The last expression is equal to the first while the others are all different; we omit the last expression and have n distinct values each being an nth root of $A\,e^{i\theta}$. As examples the three cube roots of unity are obtained from $e^{i0} = e^{i2\pi} = e^{i4\pi}$ as e^{i0}, $e^{i(2\pi/3)}$, $e^{i(4\pi/3)}$. The two square roots of $4 \text{ cis } 20°$ are $2 \text{ cis } 10°$ and $2 \text{ cis } 190°$.

4.10 Complex Numbers Raised to Complex Powers. To raise the complex number $A \text{ cis } \theta$ to the complex power $a + ib$ (where a and b are real integers) we have

$$(A\,e^{i\theta})^{a+ib} = A^a A^{ib}\,e^{ia\theta} e^{-b\theta}. \tag{4.12}$$

From the definition of the logarithm we have

$$A = e^{\ln A}. \tag{4.13}$$

Therefore

$$A^{ib} = e^{ib \ln A} \tag{4.14}$$

and

$$(A \text{ cis } \theta)^{a+ib} = A^a e^{-b\theta} e^{i(a\theta + b \ln A)} \tag{4.15}$$

$$= A^a e^{-b\theta} \text{cis}(a\theta + b \ln A). \tag{4.16}$$

In this equation $b \ln A$ is in radians, and θ must be measured in radians in $e^{-b\theta}$.

4.11 Complex Numbers Raised to Rational Powers. This is a combination of the processes of raising to integer powers and extracting integer roots both of which have been described.

$$(A \text{ cis } \theta)^{1.3} = [(A \text{ cis } \theta)^{13}]^{\frac{1}{10}}. \tag{4.17}$$

Raising to the 13th power is done as described in section 4.8 and the 10th root is extracted according to the directions in sections 4.9. Note that $(A \text{ cis } \theta)^{1.3}$ has ten values whereas $(A \text{ cis } \theta)^{1.4}$ has only five values.

4.12 Conjugates. The conjugate of the complex number $a + ib$ is defined as $a - ib$. Note that the real parts of the number and its conjugate are the same and that the imaginary parts differ in sign. Therefore the conjugate of $a - ib$ is $a + ib$.

If we refer to the trigonometric form of the complex number we have

$$a + ib = A \text{ cis } \theta = A \ (\cos \theta + i \sin \theta). \tag{4.18}$$

Therefore

$$a - ib = A \ (\cos \theta - i \sin \theta) = A \ (\cos -\theta + i \sin -\theta) = A \text{ cis } -\theta. \tag{4.19}$$

We see that the conjugate of A cis θ is A cis $-\theta$. A complex number and its conjugate both have the same modulus and their arguments are numerically equal but differ in sign.

The following theorems concerning complex numbers and their conjugates are easily proved from the definition above.

A. The conjugate of the sum of several complex numbers is equal to the sum of the conjugates of the complex numbers.

B. The conjugate of the product of several complex numbers is equal to the product of the conjugates of the complex numbers.

C. The conjugate of the quotient of two complex numbers is equal to the quotient of their conjugates.

D. A necessary and sufficient condition that a number be equal to its conjugate is that it be a real number.

E. A necessary and sufficient condition at the sum of a complex number and its conjugate be zero is that it be an imaginary number.

F. The sum of a complex number and its conjugate is a real number.

G. The product of a complex number and its conjugate is a non-negative real number.

4.13 Logarithms of Complex Numbers. The logarithm of a complex number is found very easily if the number is written in the exponential form: $\ln (A\, e^{i\theta}) = \ln A + \ln e^{i\theta} = \ln A + i\theta$. As an example, $\ln (-1) = \ln e^{i\pi} = i\pi$. Further, $\ln 1 = \ln -1 + \ln -1 = i\pi + i\pi = i2\pi$. We often assume that the logarithm of unity to any base is zero. We now see that the natural logarithm of unity is any multiple of $i2\pi$. Therefore we can add any multiple of $i2\pi$ to a natural logarithm without changing the antilogarithm, just as we can add 360° to an angle without changing its cosine. $\cos 60° = \cos 420° = 0.5$.

* **4.14 Trigonometric Functions.** Euler's theorem $e^{ix} = \cos x + i \sin x$ becomes, when x is replaced by $-x$, $e^{-ix} = \cos x - i \sin x$. These two equations can be solved for $\cos x$ and $\sin x$ giving

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{i2} \tag{4.20}$$

and corresponding expressions can be obtained for the other trigonometric functions from

$$\begin{aligned} \tan x &= \frac{\sin x}{\cos x}, & \cot x &= \frac{\cos x}{\sin x}, \\ \sec x &= \frac{1}{\cos x}, & \csc x &= \frac{1}{\sin x}. \end{aligned} \tag{4.21}$$

These formulas are very useful for checking trigonometric transformations and in simplifying complicated expressions. For example, we can check the relation $\cos (a + b) = \cos a \cos b - \sin a \sin b$ as follows:

$$\begin{aligned} \cos a \cos b - \sin a \sin b &= \frac{e^{ia} + e^{-ia}}{2}\frac{e^{ib} + e^{-ib}}{2} - \frac{e^{ia} - e^{-ia}}{i2}\frac{e^{ib} - e^{-ib}}{i2} \\ &= \frac{e^{i(a+b)} + e^{-i(a+b)}}{2} = \cos (a + b). \end{aligned}$$

A table of the more common transformations follows. It is recommended that one or more of the formulas be checked for practice in handling these exponential expressions.

TABLE IV–1

$$\sin -x = -\sin x \qquad \csc -x = -\csc x$$
$$\cos -x = \cos x \qquad \sec -x = \sec x$$
$$\tan -x = -\tan x \qquad \cot -x = -\cot x$$
$$\cos x + i \sin x = e^{ix} \qquad \cos x - i \sin x = e^{-ix}$$
$$\cos^2 x + \sin^2 x = 1$$
$$\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b$$
$$\sin (a \pm b) = \sin a \cos b \pm \cos a \sin b$$
$$\tan (a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} \qquad \cot (a \pm b) = \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a}$$
$$\cos 2a = \cos^2 a - \sin^2 a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a$$
$$\sin 2a = 2 \sin a \cos a$$
$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a} \qquad \cot 2a = \frac{\cot^2 a - 1}{2 \cot a}$$
$$\cos \frac{a}{2} = \sqrt{\frac{1 + \cos a}{2}} \qquad \sin \frac{a}{2} = \sqrt{\frac{1 - \cos a}{2}}$$
$$\tan \frac{a}{2} = \sqrt{\frac{1 - \cos a}{1 + \cos a}} \qquad \cot \frac{a}{2} = \sqrt{\frac{1 + \cos a}{1 - \cos a}}$$
$$\sec \frac{a}{2} = \sqrt{\frac{2}{1 + \cos a}} \qquad \csc \frac{a}{2} = \sqrt{\frac{2}{1 - \cos a}}$$
$$\sin a \sin b = \tfrac{1}{2} [\cos (a - b) - \cos (a + b)]$$
$$\sin a \cos b = \tfrac{1}{2} [\sin (a + b) + \sin (a - b)]$$
$$\cos a \cos b = \tfrac{1}{2} [\cos (a - b) + \cos (a + b)]$$

4.15 Hyperbolic Functions. If we let $x = iu$ in the expression for $\cos x$ obtained in the preceding section, we have

$$\cos iu = \frac{e^u + e^{-u}}{2}. \tag{4.22}$$

The right side of the above equation is a real function of u and it seems confusing to write the imaginary unit in a real expression. The right-hand side of the above equation has been assigned a special name. Since we obtained it from the expression for the cosine we include the term cosine in the name and call it the hyperbolic cosine of u, abbreviated $\cosh u$. (The abbreviated expression is read *h*-cosine u.) We define the hyperbolic sine of u as the following expression:

$$\sinh u = \frac{e^u - e^{-u}}{2}. \tag{4.23}$$

This with the expression for the hyperbolic cosine

$$\cosh u = \frac{e^u + e^{-u}}{2} \tag{4.24}$$

enables us to obtain the remaining four hyperbolic functions from

$$\begin{aligned} \tanh u &= \frac{\sinh u}{\cosh u}, & \coth u &= \frac{\cosh u}{\sinh u}, \\ \operatorname{csch} u &= \frac{1}{\sinh u}, & \operatorname{sech} u &= \frac{1}{\cosh u}. \end{aligned} \tag{4.25}$$

Just as for a numerical value of x we refer to tables to get the numerical value of $\sin x$, etc., so for a given value of u we refer to tables of hyperbolic functions to obtain the value of $\cosh u$, etc. The variation of the several hyperbolic functions is illustrated in Fig. 4–5.

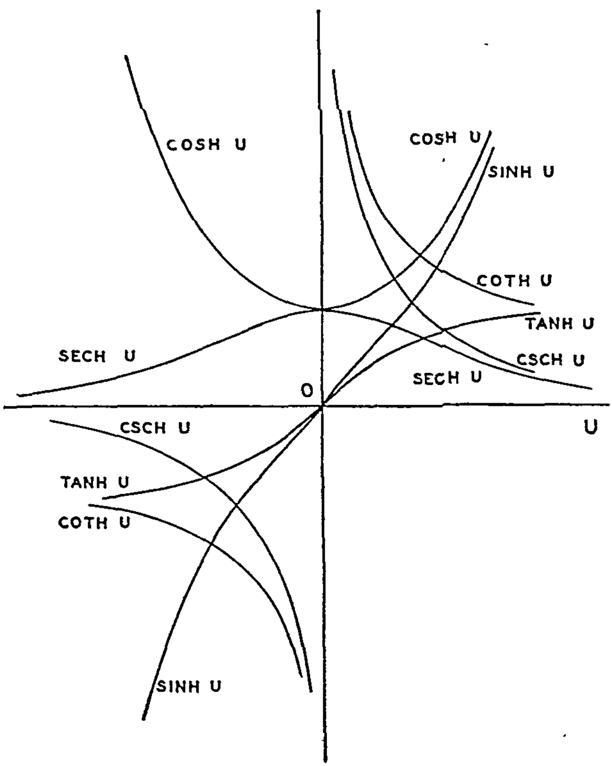

FIG. 4–5

4.16 Relations between Hyperbolic Functions and Circular Functions. Our definition of the hyperbolic cosine leads at once to the relation $\cosh u = \cos iu$. The definitions of the other hyperbolic functions enable us to find similar relations between them and the corresponding circular functions.

The various equalities are listed below. Any of them can be demonstrated from the two preceding sections.

TABLE IV–2

$\cosh u = \cos iu$	$\cos x = \cosh ix$
$\sinh u = -i \sin iu$	$\sin x = -i \sinh ix$
$\tanh u = -i \tan iu$	$\tan x = -i \tanh ix$
$\coth u = i \cot iu$	$\cot x = i \coth ix$
$\operatorname{sech} u = \sec iu$	$\sec x = \operatorname{csch} ix$
$\operatorname{csch} u = i \csc iu$	$\csc x = i \operatorname{csch} ix$

These transformations enable us to derive any relation for hyperbolic functions for which we have the corresponding relation for circular functions. For example, to find $\cosh(a + b)$

$$\begin{aligned}\cosh(a+b) &= \cos(ia+ib) = \cos ia \cos ib - \sin ia \sin ib \\ &= \cosh a \cosh b - i^2 \sinh a \sinh b \\ &= \cosh a \cosh b + \sinh a \sinh b.\end{aligned}$$

A table of such formulas follows. Any of the transformations can be checked in the manner just described or by substituting the exponential expressions for the hyperbolic functions. As an example of the latter method let us check the relation $\cosh^2 u - \sinh^2 u = 1$.

$$\begin{aligned}\cosh^2 u - \sinh^2 u &= \left(\frac{e^u + e^{-u}}{2}\right)^2 - \left(\frac{e^u - e^{-u}}{2}\right)^2 \\ &= \frac{e^{2u} + 2 + e^{-2u} - e^{2u} + 2 - e^{-2u}}{4} = 1. \quad (4.26)\end{aligned}$$

TABLE IV–3

$$\sinh -u = -\sinh u \qquad \cosh -u = \cosh u$$
$$\tanh -u = -\tanh u \qquad \coth -u = -\coth u$$
$$\operatorname{sech} -u = \operatorname{sech} u \qquad \operatorname{csch} -u = -\operatorname{csch} u$$
$$\cosh u + \sinh u = e^u \qquad \cosh u - \sinh u = e^{-u}$$
$$\cosh^2 u - \sinh^2 u = 1$$
$$\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$$
$$\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$$
$$\tanh(a \pm b) = \frac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b} \qquad \coth(a \pm b) = \frac{\coth a \coth b \pm 1}{\coth b \pm \coth a}$$
$$\cosh 2a = \cosh^2 a + \sinh^2 a = 1 + 2\sinh^2 a = 2\cosh^2 a - 1$$
$$\sinh 2a = 2 \sinh a \cosh a$$
$$\tanh 2a = \frac{2 \tanh a}{1 + \tanh^2 a} \qquad \coth 2a = \frac{\coth^2 a + 1}{2 \coth a}$$
$$\cosh \frac{a}{2} = \sqrt{\frac{\cosh a + 1}{2}} \qquad \sinh \frac{a}{2} = \sqrt{\frac{\cosh a - 1}{2}}$$

TABLE IV-3 (*Continued*)

$$\tanh\frac{a}{2} = \sqrt{\frac{\cosh a - 1}{\cosh a + 1}} \qquad \coth\frac{a}{2} = \sqrt{\frac{\cosh a + 1}{\cosh a - 1}}$$

$$\operatorname{sech}\frac{a}{2} = \sqrt{\frac{2}{\cosh a + 1}} \qquad \operatorname{csch}\frac{a}{2} = \sqrt{\frac{2}{\cosh a - 1}}$$

$$\begin{aligned} \sinh a \sinh b &= \tfrac{1}{2}[\cosh(a+b) - \cosh(a-b)] \\ \sinh a \cosh b &= \tfrac{1}{2}[\sinh(a+b) + \sinh(a-b)] \\ \cosh a \cosh b &= \tfrac{1}{2}[\cosh(a+b) + \cosh(a-b)] \end{aligned}$$

4.17 Circular and Hyperbolic Functions of Complex Numbers. We are now in a position to write the circular or hyperbolic function of a complex number. As an example, $\sin(a + ib) = \sin a \cos ib + \cos a \sin ib$ using the standard formula for the sine of the sum of two numbers. We showed above that $\cos ib = \cosh b$ and $\sin ib = i \sinh b$; therefore $\sin(a + ib) = \sin a \cosh b + i \cos a \sinh b$. Now in this expression each of the four functions on the right can be found in tables and therefore we have shown how the student can evaluate $\sin(a + ib)$ as a complex number.

The following table gives the formulas needed to obtain the functions of complex numbers. The tangent can be found by dividing the sine by the cosine, etc.

TABLE IV-4

$$\begin{aligned} \sin(a \pm ib) &= \sin a \cosh b \pm i \cos a \sinh b \\ \cos(a \pm ib) &= \cos a \cosh b \mp i \sin a \sinh b \\ \sinh(a \pm ib) &= \sinh a \cos b \pm i \cosh a \sin b \\ \cosh(a \pm ib) &= \cosh a \cos b \pm i \sinh a \sin b \end{aligned}$$

4.18 Periodicity. It will be recalled that the circular functions have a period of 2π radians or 360°, etc., except the tangent and cotangent which have as a period π radians or 180°. It is interesting to see what corresponds to this in the case of hyperbolic functions.

$$\cosh(a + i2\pi) = \cosh a \cos 2\pi + i \sinh a \sin 2\pi = \cosh a. \quad (4.27)$$

Therefore, $i2\pi$ is the period for the hyperbolic cosine. The result of this type of investigation is listed below and the corresponding relations for the circular functions are included to complete the story.

TABLE IV-5

$\cosh(a + i2\pi) = \cosh a$	$\cos(a + 2\pi) = \cos a$
$\sinh(a + i2\pi) = \sinh a$	$\sin(a + 2\pi) = \sin a$
$\tanh(a + i\pi) = \tanh a$	$\tan(a + \pi) = \tan a$
$\coth(a + i\pi) = \coth a$	$\cot(a + \pi) = \cot a$
$\operatorname{csch}(a + i2\pi) = \operatorname{csch} a$	$\csc(a + 2\pi) = \csc a$
$\operatorname{sech}(a + i2\pi) = \operatorname{sech} a$	$\sec(a + 2\pi) = \sec a$

Since the circular and hyperbolic tangent and cotangent have a period of π or $i\pi$, it is interesting to note the effect of adding $i\pi$ to the argument of the hyperbolic sine, etc.

$$\sinh (a + i\pi) = \sinh a \cos \pi + i \cosh a \sin \pi = -\sinh a. \tag{4.28}$$

The other relations can easily be found in a similar manner. The results are listed below and can be checked several ways.

TABLE IV-6

$\cosh (a \pm i\pi) = -\cosh a$	$\cos (a \pm \pi) = -\cos a$
$\sinh (a \pm i\pi) = -\sinh a$	$\sin (a \pm \pi) = -\sin a$
$\tanh (a \pm i\pi) = \tanh a$	$\tan (a \pm \pi) = \tan a$
$\coth (a \pm i\pi) = \coth a$	$\cot (a \pm \pi) = \cot a$
$\operatorname{sech} (a \pm i\pi) = -\operatorname{sech} a$	$\sec (a \pm \pi) = -\sec a$
$\operatorname{csch} (a \pm i\pi) = -\operatorname{csch} a$	$\csc (a \pm \pi) = -\csc a$

In the case of circular functions we can change from the sine to the cosine by adding 0.5π.

$$\sin \left(a + \frac{\pi}{2}\right) = \sin a \cos \frac{\pi}{2} + \cos a \sin \frac{\pi}{2} = \cos a. \tag{4.29}$$

A similar type of transformation is possible with hyperbolic functions as shown in the following table.

TABLE IV-7

$\sinh \left(a \pm i\frac{\pi}{2}\right) = \pm i \cosh a$	$\sin \left(a \pm \frac{\pi}{2}\right) = \pm \cos a$
$\cosh \left(a \pm i\frac{\pi}{2}\right) = \pm i \sinh a$	$\cos \left(a \pm \frac{\pi}{2}\right) = \mp \sin a$
$\tanh \left(a \pm i\frac{\pi}{2}\right) = \coth a$	$\tan \left(a \pm \frac{\pi}{2}\right) = -\cot a$
$\coth \left(a \pm i\frac{\pi}{2}\right) = \tanh a$	$\cot \left(a \pm \frac{\pi}{2}\right) = -\tan a$
$\operatorname{sech} \left(a \pm i\frac{\pi}{2}\right) = \mp i \operatorname{csch} a$	$\sec \left(a \pm \frac{\pi}{2}\right) = \mp \csc a$
$\operatorname{csch} \left(a \pm i\frac{\pi}{2}\right) = \mp i \operatorname{sech} a$	$\csc \left(a \pm \frac{\pi}{2}\right) = \pm \sec a$

4.19 Inverse Functions. The formulas $y = \sin x$, $z = \sinh x$, and $w = \ln x$ give y, z, and w as functions of x. It may be desirable to consider x as a function of y, z, or w. If x is to be considered a function of y we would like to write x equals something involving y. This is done formally as follows: $x = \sin^{-1} y$ or $x = \text{arc} \sin y$, where $\sin^{-1} y$ or $\text{arc} \sin y$ means the angle whose sine is y. Similarly $x = \sinh^{-1} z$ means the quantity whose

hyperbolic sine is z. These functions are called "inverse functions." The third equation above can be solved for x in terms of w according to the above scheme $x = \ln^{-1} w$ and this is called the "antilogarithm" of w. This equation can also be written $x = e^w$ which needs no further discussion.

Although $(\sin x)^2 = \sin^2 x$, nevertheless $(\sin x)^{-1}$ is entirely different from $\sin^{-1} x$; in order to avoid this possible difficulty many people prefer to write arc sin x in place of $\sin^{-1} x$. However the expression arc sinh x is not used.

There is no difficulty in finding the quantity whose cosine is 0.4, or whose hyperbolic sine is 1.2; however, to find the quantity whose sine is $0.4 - i0.7$, or even the quantity whose cosine is 4.0, involves a little more than mere reference to tables. It is convenient to have available a set of formulas to take care of the problem of finding inverse functions. We shall derive the formula for $\cosh^{-1} x$ and list the other formulas without proof.

Let $\cosh u = x$; then $u = \cosh^{-1} x$; write

$$e^u = \cosh u + \sinh u \tag{4.30}$$

$$= \cosh u + \sqrt{\cosh^2 u - 1} \tag{4.31}$$

$$= x + \sqrt{x^2 - 1}. \tag{4.32}$$

Now when we take the natural logarithm of both sides of equation (4.32) we have

$$u = \ln (x + \sqrt{x^2 - 1}) + i2n\pi = \cosh^{-1} x. \tag{4.33}$$

We found in section 4.13 that any multiple of $i2\pi$ could be added to the natural logarithm without changing the antilogarithm. The term $i2n\pi$ can also be accounted for on the basis of the periodicity of the hyperbolic cosine.

The formula given in equation (4.33) can be used where x is any complex number. We have shown how to perform each step, i.e., addition, subtraction, raising to powers, extracting roots, and finding logarithms. The process may be long but it is not too difficult.

TABLE IV–8

$$\sinh^{-1} x = \ln (x + \sqrt{x^2 + 1}) + i2n\pi$$

$$\cosh^{-1} x = \ln (x + \sqrt{x^2 - 1}) + i2n\pi$$

$$\tanh^{-1} x = 0.5 \ln \left(\frac{1 + x}{1 - x}\right) + in\pi$$

$$\coth^{-1} x = 0.5 \ln \left(\frac{x + 1}{x - 1}\right) + in\pi$$

TABLE IV-8 (*Continued*)

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + i2n\pi$$

$$\operatorname{csch}^{-1} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) + i2n\pi$$

$$\text{arc sin } x = \sin^{-1} x = i \ln (x + \sqrt{x^2-1}) + \frac{\pi}{2} + 2n\pi$$

$$\text{arc cos } x = \cos^{-1} x = i \ln (x + \sqrt{x^2-1}) + 2n\pi$$

$$\text{arc tan } x = \tan^{-1} x = i0.5 \ln\left(\frac{1-ix}{1+ix}\right) + n\pi$$

$$\text{arc cot } x = \cot^{-1} x = i0.5 \ln\left(\frac{ix+1}{ix-1}\right) + n\pi$$

$$\text{arc sec } x = \sec^{-1} x = i \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + 2n\pi$$

$$\text{arc csc } x = \csc^{-1} x = i \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \frac{\pi}{2} + 2n\pi$$

In these formulas x can be any complex number; n is any real integer.

4.20 Infinite Series. Sometimes MacLaurin's series for the hyperbolic functions are needed. The series for the hyperbolic sine and cosine can be obtained by substituting the series for the exponential function into the expressions we used to define the hyperbolic sine and cosine.

The series for the exponential function is known from the calculus to be

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} \cdots. \tag{4.34}$$

Making this substitution and collecting terms we get

$$\cosh u = \frac{e^u + e^{-u}}{2} = 1 + \frac{u^2}{2} + \frac{u^4}{4!} + \frac{u^6}{6!} + \cdots, \tag{4.35}$$

$$\sinh u = \frac{e^u - e^{-u}}{2} = u + \frac{u^3}{3!} + \frac{u^5}{5!} + \frac{u^7}{7!} + \cdots. \tag{4.36}$$

The corresponding series for the circular functions are included for comparison:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots, \tag{4.37}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots. \tag{4.38}$$

These series are useful for computing the value of the functions for small

values of u and x. As an example

$$\cosh 0.2 = 1 + \frac{0.04}{2} + \frac{0.0016}{24} + \frac{0.000064}{720} \cdots$$
$$= 1 + 0.02 + 0.000067 + 0.000000089$$
$$= 1.02007, \text{ correct to 5 places.}$$

4.21 Gudermanian Function. It is possible to express any five of the circular functions in terms of the sixth, and it is also possible to express any five of the hyperbolic functions in terms of the sixth. These relations are listed in Table IV–9. Let us compare the relations we get if we express the circular functions in terms of $\sin x$ and the hyperbolic functions in terms of $\tanh u$.

$$\operatorname{sech} u = \sqrt{1 - \tanh^2 u} \qquad \cos x = \sqrt{1 - \sin^2 x}$$

$$\operatorname{csch} u = \frac{\sqrt{1 - \tanh^2 u}}{\tanh u} \qquad \cot x = \frac{\sqrt{1 - \sin^2 x}}{\sin x}$$

$$\sinh u = \frac{\tanh u}{\sqrt{1 - \tanh^2 u}} \qquad \tan x = \frac{\sin x}{\sqrt{1 - \sin^2 x}}$$

$$\coth = \frac{1}{\tanh u} \qquad \csc x = \frac{1}{\sin x}$$

$$\cosh u = \frac{1}{\sqrt{1 - \tanh^2 u}} \qquad \sec x = \frac{1}{\sqrt{1 - \sin^2 x}}$$

A comparison of the relations in the two columns above shows that if we choose x so that $\sin x = \tanh u$, then

$$\begin{aligned} \sin x &= \tanh u \\ \tan x &= \sinh u \\ \cos x &= \operatorname{sech} u \\ \sec x &= \cosh u \\ \cot x &= \operatorname{csch} u \\ \csc x &= \coth u \end{aligned}$$

If x is chosen as described above (to make $\sin x = \tanh u$), then x is called the Gudermanian of u, written $x = \operatorname{gd} u$, and u is the anti-gudermanian of x, $u = \operatorname{gd}^{-1} x$.

The Gudermanian enables us to make use of tables of circular functions when we are working with hyperbolic functions, e.g., in section 4.20 we computed the value of $\cosh 0.2 = 1.02007$. We can now find $\sinh 0.2$ by using a table of circular functions and finding $\tan x$ corresponding to $\sec x = 1.02007$ and find $\sinh 0.2 = 0.20133$.

TABLE IV-9

Relations among Circular Functions

$$\sin x = \sqrt{1-\cos^2 x} = \frac{\tan x}{\sqrt{1+\tan^2 x}} = \frac{1}{\sqrt{1+\cot^2 x}} = \frac{\sqrt{\sec^2 x - 1}}{\sec x} = \frac{1}{\csc x}$$

$$\sqrt{1-\sin^2 x} = \cos x = \frac{1}{\sqrt{1+\tan^2 x}} = \frac{\cot x}{\sqrt{1+\cot^2 x}} = \frac{1}{\sec x} = \frac{\sqrt{\csc^2 x - 1}}{\csc x}$$

$$\frac{\sin x}{\sqrt{1-\sin^2 x}} = \frac{\sqrt{1-\cos^2 x}}{\cos x} = \tan x = \frac{1}{\cot x} = \sqrt{\sec^2 x - 1} = \frac{1}{\sqrt{\csc^2 x - 1}}$$

$$\frac{\sqrt{1-\sin^2 x}}{\sin x} = \frac{\cos x}{\sqrt{1-\cos^2 x}} = \frac{1}{\tan x} = \cot x = \frac{1}{\sqrt{\sec^2 x - 1}} = \sqrt{\csc^2 x - 1}$$

$$\frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\cos x} = \sqrt{1+\tan^2 x} = \frac{\sqrt{1+\cot^2 x}}{\cot x} = \sec x = \frac{\csc x}{\sqrt{\csc^2 x - 1}}$$

$$\frac{1}{\sin x} = \frac{1}{\sqrt{1-\cos^2 x}} = \frac{\sqrt{1+\tan^2 x}}{\tan x} = \sqrt{1+\cot^2 x} = \frac{\sec x}{\sqrt{\sec^2 x - 1}} = \csc x$$

Relations among Hyperbolic Functions

$$\sinh x = \sqrt{\cosh^2 x - 1} = \frac{\tanh x}{\sqrt{1-\tanh^2 x}} = \frac{1}{\sqrt{\coth^2 x - 1}} = \frac{\sqrt{1-\operatorname{sech}^2 x}}{\operatorname{sech} x}$$
$$= \frac{1}{\operatorname{csch} x}$$

$$\sqrt{1+\sinh^2 x} = \cosh x = \frac{1}{\sqrt{1-\tanh^2 x}} = \frac{\coth x}{\sqrt{\coth^2 x - 1}} = \frac{1}{\operatorname{sech} x} = \frac{\sqrt{1+\operatorname{csch}^2 x}}{\operatorname{csch} x}$$

$$\frac{\sinh x}{\sqrt{1+\sinh^2 x}} = \frac{\sqrt{\cosh^2 x - 1}}{\cosh x} = \tanh x = \frac{1}{\coth x} = \sqrt{1-\operatorname{sech}^2 x} = \frac{1}{\sqrt{1+\operatorname{csch}^2 x}}$$

$$\frac{\sqrt{1+\sinh^2 x}}{\sinh x} = \frac{\cosh x}{\sqrt{\cosh^2 x - 1}} = \frac{1}{\tanh x} = \coth x = \frac{1}{\sqrt{1-\operatorname{sech}^2 x}} = \sqrt{1+\operatorname{csch}^2 x}$$

$$\frac{1}{\sqrt{1+\sinh^2 x}} = \frac{1}{\cosh x} = \sqrt{1-\tanh^2 x} = \frac{\sqrt{\coth^2 x - 1}}{\coth x} = \operatorname{sech} x = \frac{\operatorname{csch} x}{\sqrt{1+\operatorname{csch}^2 x}}$$

$$\frac{1}{\sinh x} = \frac{1}{\sqrt{\cosh^2 x - 1}} = \frac{\sqrt{1-\tanh^2 x}}{\tanh x} = \sqrt{\coth^2 x - 1} = \frac{\operatorname{sech} x}{\sqrt{1-\operatorname{sech}^2 x}}$$
$$= \operatorname{csch} x$$

4.22 Differentiation. Formulas for the differentiation of hyperbolic functions can be set up easily from the exponential expressions, e.g.,

$$\frac{d\cosh u}{du} = \frac{d}{du}\,\frac{e^u + e^{-u}}{2} = \frac{e^u - e^{-u}}{2} = \sinh u, \tag{4.39}$$

$$\frac{d \sinh u}{du} = \frac{d}{du} \frac{e^u - e^{-u}}{2} = \frac{e^u + e^{-u}}{2} = \cosh u. \tag{4.40}$$

For the nth derivative we have

$$\begin{aligned}
\frac{d^n \cosh u}{du^n} &= \sinh u, \quad \text{if } n \text{ is odd,} \\
\frac{d^n \cosh u}{du^n} &= \cosh u, \quad \text{if } n \text{ is even,} \\
\frac{d^n \sinh u}{du^n} &= \cosh u, \quad \text{if } n \text{ is odd,} \\
\frac{d^n \sinh u}{du^n} &= \sinh u, \quad \text{if } n \text{ is even.}
\end{aligned} \tag{4.41}$$

Table IV–10 gives the derivatives of both circular and hyperbolic functions, and Table IV–11 gives integral formulas.

TABLE IV–10

$\frac{d \sinh u}{du} = \cosh u$	$\frac{d \sin v}{dv} = \cos v$
$\frac{d \cosh u}{du} = \sinh u$	$\frac{d \cos v}{dv} = -\sin v$
$\frac{d \tanh u}{du} = \operatorname{sech}^2 u$	$\frac{d \tan v}{dv} = \sec^2 v$
$\frac{d \coth u}{du} = -\operatorname{csch}^2 u$	$\frac{d \cot v}{dv} = -\csc^2 v$
$\frac{d \operatorname{sech} u}{du} = -\operatorname{sech}^2 u \sinh u$	$\frac{d \sec v}{dv} = \sec^2 v \sin v$
$\frac{d \operatorname{csch} u}{du} = -\operatorname{csch}^2 u \cosh u$	$\frac{d \csc v}{dv} = -\csc^2 v \cos v$
$\frac{d \operatorname{gd} u}{du} = \operatorname{sech} u$	$\frac{d \operatorname{gd}^{-1} v}{dv} = \sec v$
$\frac{d \sinh^{-1} u}{du} = \frac{1}{\sqrt{u^2 + 1}}$	$\frac{d \arcsin v}{dv} = \frac{d \sin^{-1} v}{dv} = \frac{1}{\sqrt{1 - v^2}}$
$\frac{d \cosh^{-1} u}{du} = \frac{1}{\sqrt{u^2 - 1}}$	$\frac{d \arccos v}{dv} = \frac{d \cos^{-1} v}{dv} = \frac{1}{\sqrt{1 - v^2}}$
$\frac{d \tanh^{-1} u}{du} = \frac{1}{1 - u^2}$	$\frac{d \arctan v}{dv} = \frac{d \tan^{-1} v}{dv} = \frac{1}{1 + v^2}$

TABLE IV-10 (*Continued*)

$$\frac{d\coth^{-1}u}{du} = \frac{1}{1-u^2} \qquad \frac{d\,\mathrm{arc}\cot v}{dv} = \frac{d\cot^{-1}v}{dv} = \frac{-1}{1+v^2}$$

$$\frac{d\,\mathrm{sech}^{-1}u}{du} = \frac{-1}{u\sqrt{1-u^2}} \qquad \frac{d\,\mathrm{arc}\sec v}{dv} = \frac{d\sec^{-1}v}{dv} = \frac{1}{v\sqrt{v^2-1}}$$

$$\frac{d\,\mathrm{csch}^{-1}u}{du} = \frac{-1}{u\sqrt{1+u^2}} \qquad \frac{d\,\mathrm{arc}\csc v}{dv} = \frac{d\csc^{-1}v}{dv} = \frac{-1}{v\sqrt{v^2-1}}$$

TABLE IV-11

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = -\ln\cos x + C$$

$$\int \cot x\,dx = \ln\sin x + C$$

$$\int \sec x\,dx = \ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + C = \mathrm{gd}^{-1}x + C$$

$$\int \csc x\,dx = \ln\tan\frac{x}{2} + C$$

$$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$$

$$\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2} + C$$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - 0.5\ln(1+x^2) + C$$

$$\int \cot^{-1}x\,dx = x\cot^{-1}x + 0.5\ln(1+x^2) + C$$

$$\int \sec^{-1}x\,dx = x\sec^{-1}x - \cosh^{-1}x + C$$

$$\int \csc^{-1}x\,dx = x\csc^{-1}x + \cosh^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C$$

$$\int \cosh x\,dx = \sinh x + C$$

$$\int \tanh x\,dx = \ln\cosh x + C$$

TABLE IV–11 (*Continued*)

$$\int \coth x\, dx = \ln \sinh x + C$$

$$\int \operatorname{sech} x\, dx = 2 \tan^{-1} (e^x) + C = \operatorname{gd} x + C$$

$$\int \operatorname{csch} x\, dx = \ln \tanh \frac{x}{2} + C = -2 \tanh^{-1} (e^x) + C$$

$$\int \sinh^{-1} x\, dx = x \sinh^{-1} x - \sqrt{x^2 + 1} + C$$

$$\int \cosh^{-1} x\, dx = x \cosh^{-1} x - \sqrt{x^2 - 1} + C$$

$$\int \tanh^{-1} x\, dx = x \tanh^{-1} x + 0.5 \ln (1 - x^2) + C$$

$$\int \coth^{-1} x\, dx = x \coth^{-1} x + 0.5 \ln (x^2 - 1) + C$$

$$\int \operatorname{sech}^{-1} x\, dx = x \operatorname{sech}^{-1} x - \cos^{-1} x + C$$

$$\int \operatorname{csch}^{-1} x\, dx = x \operatorname{csch}^{-1} x + \sinh^{-1} x + C$$

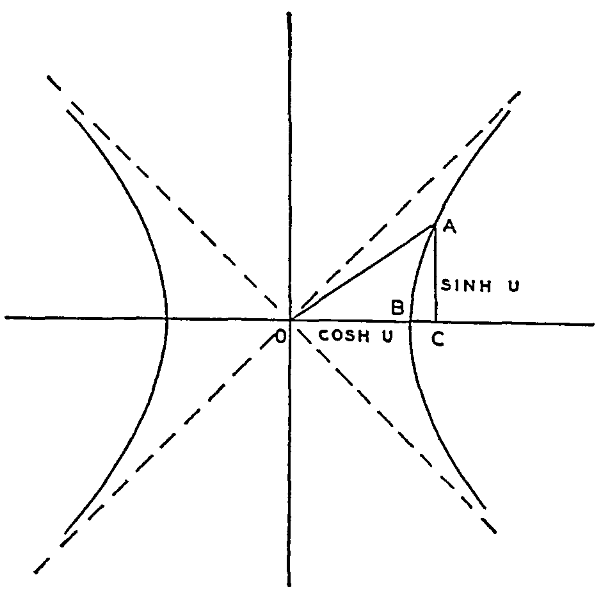

FIG. 4–6

4.23 Geometric Interpretation of Hyperbolic Functions. The reason for calling these functions hyperbolic functions is often asked by the stu-

dent. The connection between these functions and the hyperbola is interesting but not particularly important for most purposes. The relation $\cosh^2 u = \sinh^2 u + 1$ is the equation of a hyperbola (see Fig. 4–6), where $\cosh u$ is plotted along the horizontal axis and $\sinh u$ is plotted along the vertical axis. We should not jump to the conclusion at this point that u is the angle AOB in the figure. This cannot be since $\tan AOB = \sinh u/\cosh u = \tanh u$. The relation between u and the geometric figure is that u is equal to twice the area of the sector $OABO$. This can be shown as follows:

$$\begin{aligned}\text{area of sector} &= \text{area of triangle } OAC - \text{area } BAC \\ &= 0.5 \cosh u \sinh u - \int_1^{\cosh u} y\, dx.\end{aligned}$$

where a set of x- and y-axes have been placed on the original axes to simplify the integration.

$$\begin{aligned}\text{Area of sector} &= 0.25 \sinh 2u - \int_0^u \sinh u\, d\,(\cosh u) \\ &= 0.25 \sinh 2u - \int_0^u \sinh^2 u\, du \\ &= 0.25 \sinh 2u - \int_0^u 0.5\,(\cosh 2u - 1)\, du \\ &= 0.25 \sinh 2u - \left[0.25 \sinh 2u - \frac{u}{2}\right]_0^u \\ &= 0.25 \sinh 2u - 0.25 \sinh 2u + \frac{u}{2} \\ &= \frac{u}{2},\end{aligned}$$

or u is double the area of the sector.

In the case of circular functions $\cos^2 x + \sin^2 x = 1$ is the equation of a circle; see Fig. 4–7. The angle x is equal to twice the area of the sector $OACO$; this can be shown readily by the student.

Another way that u could have been expressed in Fig. 4–6 is by the relation

$$u = \int_B^A \frac{ds}{r}$$

where ds is an element of length of the arc and r is the distance of ds from

the origin. Using the same x and y axes we have

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2}\, du$$

$$r = \sqrt{x^2 + y^2}.$$

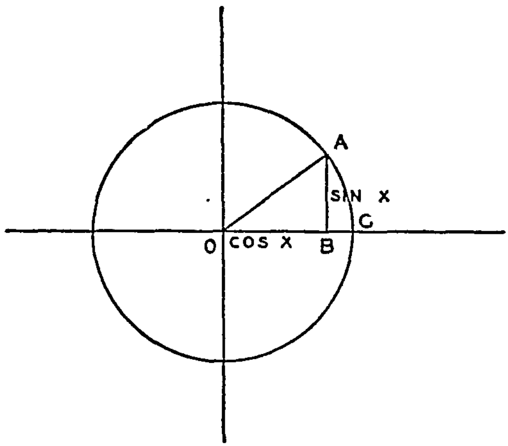

FIG. 4-7

Now since $x = \cosh u$ and $y = \sinh u$, $dx/du = \sinh u$ and $dy/du = \cosh u$ so that

$$ds = \sqrt{\sinh^2 u + \cosh^2 u}\, du,$$

$$r = \sqrt{\cosh^2 u + \sinh^2 u}.$$

Therefore,

$$\int_B^A \frac{ds}{r} = \int_0^u \frac{\sqrt{\sinh^2 u + \cosh^2 u}}{\sqrt{\cosh^2 u + \sinh^2 u}}\, du = \int_0^u du = u.$$

In the case of the circle

$$\int_C^A \frac{ds}{r} = \int_C^A ds = \text{length of arc.}$$

PROBLEMS ON CHAPTER 4

1. Change the following complex numbers to the trigonometric form:

(a)	$2 + i3$	(e)	$-3 - i2$	(i)	$5.42 + i6.37$	(m)	$-20 - i1.8$
(b)	$3 + i2$	(f)	$-8 + i5$	(j)	$5.47 + i4.92$	(n)	$-1 - i20$
(c)	$3 - i2$	(g)	$10 + i0$	(k)	$7.81 - i1.47$	(p)	$2 - i15$
(d)	$-3 + i2$	(h)	$-10 - i9$	(l)	$0 \quad - i20$	(q)	$-2 - i15$

2. Change the following complex numbers to the orthogonal form:

(a) $2 \text{ cis } 20°$	(e) $7 \text{ cis } 160°$	(i) $5 \text{ cis } -160°$	(m) $2 \text{ cis } 90°$
(b) $5 \text{ cis } 60°$	(f) $9 \text{ cis } -80°$	(j) $6 \text{ cis } 180°$	(n) $8 \text{ cis } -110°$
(c) $3 \text{ cis } 5°$	(g) $3 \text{ cis } -20°$	(k) $8 \text{ cis } -100°$	(p) $8 \text{ cis } 80°$
(d) $6 \text{ cis } 110°$	(h) $5 \text{ cis } 200°$	(l) $3 \text{ cis } 175°$	(q) $2 \text{ cis } -90°$

Evaluate the following:

3. $(2 + i3) + (4 + i5)$
4. $(2 + i3) + (4 - i5)$
5. $(-2 - i6) + (3 + i6)$
6. $(9 + i2) + (6 - i2)$
7. $(4 + i5) - (6 - i2)$
8. $(6 - i3) - (9 - i2)$
9. $(6.24 - i5.36) + (7.94 + i8.52)$
10. $(5.89 + i2.34) - (9.82 - i3.45)$
11. $(2.15 - i3.58) - (2.11 + i6.35)$
12. $(9.11 - i2.11) - (4.32 - i7.22)$
13. $2 \text{ cis } 10° + 5 \text{ cis } 20°$
14. $3 \text{ cis } 30° + 8 \text{ cis } 111°$
15. $7.2 \text{ cis } 40° + 3.2 \text{ cis } 200°$
16. $3.1 \text{ cis } 300° + 4.1 \text{ cis } 100°$
17. $5 \text{ cis } 90° - 8 \text{ cis } 330°$
18. $4.2 \text{ cis } -300° - 2.5 \text{ cis } 10°$
19. $5.3 \text{ cis } 25° - 8.3 \text{ cis } 75°$
20. $6.5 \text{ cis } 60° - 4.8 \text{ cis } 10°$
21. $(3 \text{ cis } 10°)(5 \text{ cis } 20°)$
22. $(9 \text{ cis } 30°)(9 \text{ cis } 300°)$
23. $(4 \text{ cis } 60°)(6 \text{ cis } 120°)$
24. $(5 \text{ cis } 30°)(5 \text{ cis } 15°)$
25. $(2.1 + i3.8)(2.1 - i3.8)$
26. $(3.8 - i4.4)(-2.8 - i3.4)$
27. $(8.32 - i4.83)(-2.14 - i3.24)$
28. $(-5.1 - i2.3)(2.1 - i3.1)$

Evaluate the following quotients:

29. $\dfrac{3 \text{ cis } 5°}{2 \text{ cis } 10°}$
30. $\dfrac{5 \text{ cis } 50°}{2 \text{ cis } 100°}$
31. $\dfrac{8 \text{ cis } 100°}{4 \text{ cis } 50°}$
32. $\dfrac{8.14 \text{ cis } 27°}{4.23 \text{ cis } 47°}$
33. $\dfrac{2 + i3}{2 - i3}$
34. $\dfrac{3.46 - i2.53}{-2.11 + i3.42}$
35. $\dfrac{-8.14 + i2.16}{-2.11 - i3.25}$
36. $\dfrac{-5.25 - i7.94}{2.15 - i3.50}$

Evaluate the following expressions:

37. $(2 + i3)^2$
38. $(3 - i3)^3$
39. $(-2 + i2)^2$
40. $(-2 - i3)^4$
41. $(2 + i3)^{.5}$
42. $(3 + i3)^{.25}$
43. $(-2 + i2)^{.6}$
44. $(-2 - i3)^{1.4}$
45. $(2 + i3)^{2+i2}$
46. $(3 + i3)^{2-i3}$
47. $(-2 + i2)^{1-i2}$
48. $(-2 - i3)^{1-i}$
49. $(2 + i3)^{-1+i}$
50. $(3 + i)^{2+i}$
51. $(1 + i)^{1+i}$
52. $(2 + i)^{i}$
53. $4^{1.2}$
54. $(-2)^{.3}$
55. $1^{1.6}$
56. $(-1)^{2.4}$
57. $(1 - i)^{.5}$
58. $(-2 - i3)^{1.2}$
59. $(1 + i2)^{i1.2}$
60. $(3 - i2)^{1.2+i1.4}$

61. Prove theorem A of section 4.12.
62. Prove theorem B of section 4.12.
63. Prove theorems C and D of section 4.12.
64. Prove theorem E of section 4.12.
65. Prove theorems F and G of section 4.12.

Find the natural logarithms of the following numbers:

66. $2 + i2$
67. $5 + i2$
68. $i4$
69. $-3 - i2$
70. $-5 - i13$
71. $1 - i20$
72. $6 - i5$
73. $5 - i6$

Find the numbers whose natural logarithms are:

74. 5
75. $i5$
76. $2 - i3$
77. $9 + i7$
78. π
79. $i\pi$

Prove the following formulas:

80. $\sin x = -\sin -x$
81. $\cos -x = \cos x$
82. $\cos^2 x + \sin^2 x = 1$
83. $\cos 2a = \cos^2 x - \sin^2 a$
84. $\tan 2a = \dfrac{2 \tan a}{1 + \tan^2 a}$
85. $\sin \dfrac{a}{2} = \sqrt{\dfrac{1 - \cos a}{2}}$
86. $\sin a \sin b = \frac{1}{2}[\cos (a - b) - \cos (a + b)]$
87. Prove all the formulas in Table IV–1.

Find the following (use tables or slide rule, etc.)

88. $\sin 5°$
89. $\sin 0.5$ rad
90. $\tan 0.16$ rad
91. $\sec 0.02$ rad
92. $\sinh 0.5$
93. $\cosh 0.19$
94. $\tanh 0.24$
95. $\operatorname{csch} 2.2$
96. Prove the formulas in Table IV–2.

Prove the following formulas:

97. $\sinh -u = -\sinh u$
98. $\cosh -u = \cosh u$
99. $\cosh^2 u - \sinh^2 u = 1$
100. $\tanh (a \pm b) = \dfrac{\tanh a \pm \tanh b}{1 \pm \tanh a \tanh b}$
101. $\sinh 2a = 2 \sinh a \cosh a$
102. $\operatorname{sech} \dfrac{a}{2} = \sqrt{\dfrac{2}{\cosh a - 1}}$
103. $\sinh a \sinh b = \frac{1}{2}[\cosh (a + b) - \cosh (a - b)]$
104. Prove the formulas in Table IV–3.
105. Prove the formulas in Table IV–4.

Evaluate the following in orthogonal form:

106. $\sin (2 + i0.3)$
107. $\cos (3 - i0.3)$
108. $\cosh (0.1 - i0.2)$
109. $\sinh (-0.2 - i0.3)$
110. $\sinh (-0.3 + i0.1)$
111. $\cosh (-0.2 - i0.6)$
112. $\tan (3 + i0.1)$
113. $\tanh (0.1 - i0.3)$
114. Prove the formulas in Table IV–5.
115. Prove the formulas in Table IV–6.
116. Prove the formulas in Table IV–7.
117. Find the quantity whose hyperbolic sine is:
$0.3, \quad 1.2, \quad i2, \quad 1 - i3, \quad -2 - i5$
118. Find the quantity whose hyperbolic cosine is:
$0.3, \quad 1.2, \quad i2, \quad 1 - i3, \quad -2 - i5$
119. Find the quantity whose hyperbolic tangent is:
$0.3, \quad 1.2, \quad i2, \quad 1 - i3, \quad -2 - i5$
120. Prove the formulas in Table IV–8.

121. Show that MacLaurin's series for cosh u as given in the text is correct.
122. Show that MacLaurin's series for sinh u as given in the text is correct.
123. Check the formulas in Table IV–9.
124. Check the formulas in Table IV–10.
125. Check the formulas in Table IV–11.
126. Prove $\tan x > x > \sin x$ for x real between 0 and $\pi/2$.
127. Prove $\sinh u > u > \tanh u$ for u a real positive number.

CHAPTER 5

ALGEBRAIC EQUATIONS

5.1 Introduction. The solution of many engineering problems depends upon the solutions of one or more algebraic equations. Many problems involving beam deflections, electric circuits, and mechanical vibrations lead to a certain type of differential equation known as a linear differential equation with constant coefficients. One of the first steps in the solution of such an equation is the solution of an algebraic equation that may be of the sixth or eighth or even twenty-seventh degree! It is therefore essential that the student know something about the theory and solution of algebraic equations in general.

The solutions of linear and quadratic equations are taught the student in the early courses in algebra. The solutions of equations of the third and fourth degree can be obtained with varying degrees of accuracy by using formulas to be found in most handbooks. To obtain solutions of equations of higher degree than the fourth it becomes necessary to apply methods that are not found in many handbooks. The application of the process requires an understanding of some of the fundamentals of the theory of equations.

In this study we shall be concerned with equations of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{5.1}$$

in which $a_0, a_1 \cdots a_n$ are real constants. The left member of the above equation is called a polynomial in x of degree n and often will be written $f(x)$ for brevity.

***5.2 Fundamental Theorem of Algebra.** Every equation of the form given above has a root. That is, there is a value r which may be a real, imaginary, or other complex number such that, if it be substituted for x in the polynomial, the polynomial will become equal to zero. (This theorem is assumed without proof in most algebra textbooks. The reader is referred to a text on Functions of a Complex Variable for its proof.)

***5.3 Theorem.** If $(x - r)$ is a factor of $f(x)$, then r is a root of the equation $f(x) = 0$. If $(x - r)$ is a factor of $f(x)$, then $f(x)$ can be written in the form $f(x) \equiv (x - r)Q(x)$ and it is at once evident that r is a root since

$$f(r) \equiv (r - r)Q(r) = 0. \tag{5.2}$$

***5.4 Remainder Theorem.** If the polynomial $f(x)$ be divided by $(x - r)$, the remainder will be $f(r)$. Let $f(x)$ be divided by $(x - r)$; call the quotient, which is a polynomial in x, $Q(x)$; and designate the remainder by R. Then we have

$$f(x) \equiv (x - r)Q(x) + R. \tag{5.3}$$

This is an identity in x, that is, the equation is true for any value of x whatsoever. Then the equation is true for $x = r$; substituting $x = r$ we get

$$f(r) = (r - r)Q(r) + R, \tag{5.4}$$

$$f(r) = R, \tag{5.5}$$

***5.5 Factor Theorem.** If r is a root of the equation $f(x) = 0$, then $(x - r)$ is a factor of the polynomial $f(x)$. If $f(x)$ is divided by $(x - r)$ the remainder is $f(r)$.

Therefore

$$f(x) \equiv (x - r)Q(x) + f(r). \tag{5.6}$$

If r is a root of $f(x) = 0$, then $f(r) = 0$ and we have

$$f(x) \equiv (x - r)Q(x). \tag{5.7}$$

***5.6 Theorem.** A polynomial equation of degree n cannot have more than n roots.

Consider the equation $f(x) = 0$. By the theorem of (5.2), $f(x) = 0$ has a root; call this root r_1 and the factor theorem enables us to write

$$f(x) \equiv (x - r_1)Q_1(x). \tag{5.8}$$

$Q_1(x)$ is a polynomial of degree one less than $f(x)$ and by the theorem of (5.2) $Q_1(x) = 0$ has a root which we shall designate as r_2 and the factor theorem applied to $Q_1(x) = 0$ gives us

$$Q_1(x) \equiv (x - r_2)Q_2(x), \tag{5.9}$$

$$f(x) \equiv (x - r_1)(x - r_2)Q_2(x). \tag{5.10}$$

$Q_2(x)$ has a root which we call r_3 and application of the factor theorem determines a Q_3, etc. This process can be continued until the last Q obtained is of the first degree. This gives n factors containing x

$$f(x) \equiv (x - r_1)(x - r_2) \cdots (x - r_n)a_0. \tag{5.11}$$

It is evident in the factored form above that, if r_1, r_2, etc., are all different, this set of n numbers will satisfy the equation.

No other numbers will satisfy the equation. This can be shown as follows. Assume that r_p is a root of $f(x) = 0$ which differs from those above. Substitution of $x = r_p$ in equation (5.11) would make the left

side zero but the right side would be the product of a finite number of non-zero factors and therefore would differ from zero. This of course is impossible since equation (5.11) is an identity in x, true for every value of x. This therefore proves that there are no more than n numbers which will satisfy an equation of the nth degree.

In the factored form of the equation, it sometimes happens that several factors are equal, e.g., perhaps $r_1 = r_2$, $r_3 = r_5$, etc. In this case the equation of the nth degree has less than n distinct roots. If $r_1 = r_2$ and r_1 is not equal to any other value of r, then r_1 is called a double root; similarly, we can define a triple root, etc., and, in general, if m factors are alike, we have an m-fold root.

An equation of the nth degree has n roots if we count double roots twice, triple roots three times, etc.

5.7 Theorem. If a polynomial in x of degree n equals zero for more than n distinct values of x, it is zero for all values of x.

This theorem follows from the preceding theorem. If the polynomial is of degree n there is no term containing a higher power of x than x^n. If the coefficient of x^n is not zero the polynomial can be zero for only n values of x. Therefore the coefficient of x^n must be zero. If the coefficient of x^{n-1} is not zero the polynomial can be zero for only $n - 1$ values of x. Therefore the coefficient of x^{n-1} must be zero. Similarly the coefficient of every term containing x must be zero. Now if the constant term is not zero the polynomial will not be zero for any value of x. Therefore the constant term is zero and the polynomial is zero for every value of x.

5.8 Theorem. A necessary and sufficient condition that a polynomial vanish identically is that all its coefficients be zero.

If all the coefficients are zero it is evident that this is sufficient to make the polynomial vanish identically. That this is also a necessary condition is evident from the fact that if the polynomial is of degree n and vanishes identically we can specify $n + 1$ values of x for which the polynomial vanishes and, by the discussion of the preceding theorem, the coefficients are all zero.

5.9 Theorem. A necessary and sufficient condition that two polynomials be identical is that corresponding coefficients be equal.

Let the two polynomials be

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \tag{5.12}$$

$$b_0x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n. \tag{5.13}$$

It is apparent that $a_0 = b_0$, $a_1 = b_1$, $\cdots$ $a_n = b_n$ will make the polynomials identical and this is therefore a sufficient condition. To show that it is

also a necessary condition subtract one polynomial from the other

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_{n-1} - b_{n-1})x + (a_n - b_n). \quad (5.14)$$

If the two polynomials are identical they are equal for all values of x, and their difference is zero for all values of x. Therefore, by the preceding theorem, all the coefficients are zero and we have

$$a_0 = b_0, \quad a_1 = b_1, \quad \cdots a_n = b_n. \quad (5.15)$$

5.10 Continuity of a Polynomial. A function of x is said to be continuous at $x = a$ if $f(a)$ exists and

$$\lim_{x \to a} f(x) = f(a).$$

If a function is continuous at every point of an interval, it is said to be continuous throughout the interval.

If $f(x)$ and $g(x)$ are both continuous throughout an interval, their sum is continuous throughout the interval.

If $f(x)$ and $g(x)$ are both continuous throughout an interval, their product is continuous throughout the interval.

The function $f(x) = x$ is continuous everywhere. The function $f(x) = x^p$, where p is any non-negative integer, is continuous everywhere. A polynomial which is a sum of a finite number of continuous functions of the form $a_p x^p$ is continuous everywhere.

5.11 Theorem. If $f(x)$ is a polynomial and a and b are real numbers such that $f(a) > 0$ and $f(b) < 0$, then $f(x) = 0$ has a root between a and b.

This theorem depends on the fact that a polynomial is a continuous single-valued function. No proof is given here, but it is suggested that the student consider the geometric significance of the theorem.

5.12 Theorem. Every polynomial equation, every coefficient of which is positive, can have no positive root.

This theorem depends on the fact that the sum of a set of positive numbers cannot be zero.

5.13 Theorem. Given a polynomial $f(x)$ of degree n. There is a number X such that if $x > X$ then $f(x)$ has the same sign as a_0, in other words, for $x > X$.

$$| a_0 x^n | - | a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n | > 0. \quad (5.16)$$

Now, for $x > 0$,

$$| a_0 x^n | - | a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n | \quad (5.17)$$

$$\geqq | a_0 | x^n - | a_1 | x^{n-1} - | a_2 | x^{n-2} \cdots - | a_{n-1} | x - | a_n | \quad (5.18)$$

$$\geqq | a_0 | x^n - | a_k | x^{n-1} - | a_k | x^{n-2} \cdots - | a_k | x - | a_k | \quad (5.19)$$

where $|a_k|$ is the greatest of $|a_1|, |a_2|, \cdots |a_n|$. If $x > 1$ the last polynomial (5.19) is greater than

$$|a_0| x^n - n |a_k| x^{n-1} \tag{5.20}$$

which is larger than zero if $x > n |a_k/a_0|$. Therefore, let

$$X = 1 + \left| \frac{na_k}{a_0} \right|$$

and, when $x > X$, the polynomial has the same sign as a_0.

5.14 Theorem. Given a polynomial $f(x)$ of degree n. There is a negative number Y such that if $x < Y$ then $f(x)$ has the same sign as a_0 if n is even and the opposite sign if n is odd.

Assume n even. Substitute $x = -y$. The first term in the new polynomial is $a_0 y^n$. According to the preceding theorem there is a positive number X such that when $y > X$ the polynomial has the same sign as a_0. That is, when $-x > X$ or when $x < -X$, the polynomial has the same sign as a_0. Therefore let $Y = -X$ and the theorem is proved.

Assume n odd. Substitute $x = -y$. The first term in the new polynomial is $-a_0 y^n$. According to the theorem in the preceding section there is a positive number X such that when $y > X$ the polynomial has the same sign as $-a_0$ or, in other words, the opposite sign to that of a_0 That is, when $-x > X$ or when $x < -X$, the polynomial has the opposite sign to that of a_0. Therefore let $Y = -X$ and the theorem is proved.

5.15 Theorem. Every polynomial equation of degree n when n is odd has at least one real root whose sign is opposite to the sign of the constant term when the equation is written so that the sign of the term containing x^n is plus.

Assume that $a_0 > 0$; then, by the theorem of section 5.13, there is a positive number X such that for $x = a > X, f(a) > 0$, and by the theorem of section 5.14 there is a negative number Y such that for $x = b < Y$, $f(b) < 0$. Now, if $f(0) = a_n > 0$, then by the theorem of section 5.11 there is a root between 0 and b, and therefore negative. But, if $f(0) = a_n < 0$, then by the theorem of 5.11 there is a root between 0 and a, and therefore positive.

Note that the sign of the term containing x^n can always be made plus by multiplying the equation by -1 if it is negative to begin with.

5.16 Theorem. Every polynomial equation of degree n, where n is even, having the sign of the constant term differing from the sign of the term containing x^n has at least one positive root and at least one negative root.

Make the sign of the term containing x^n positive by multiplying through by -1 if necessary. Then $f(0) = a_n < 0$. By the theorem of

also a necessary condition subtract one polynomial from the other

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_{n-1} - b_{n-1})x + (a_n - b_n). \quad (5.14)$$

If the two polynomials are identical they are equal for all values of x, and their difference is zero for all values of x. Therefore, by the preceding theorem, all the coefficients are zero and we have

$$a_0 = b_0, \quad a_1 = b_1, \quad \cdots a_n = b_n. \quad (5.15)$$

5.10 Continuity of a Polynomial. A function of x is said to be continuous at $x = a$ if $f(a)$ exists and

$$\lim_{x \to a} f(x) = f(a).$$

If a function is continuous at every point of an interval, it is said to be continuous throughout the interval.

If $f(x)$ and $g(x)$ are both continuous throughout an interval, their sum is continuous throughout the interval.

If $f(x)$ and $g(x)$ are both continuous throughout an interval, their product is continuous throughout the interval.

The function $f(x) = x$ is continuous everywhere. The function $f(x) = x^p$, where p is any non-negative integer, is continuous everywhere. A polynomial which is a sum of a finite number of continuous functions of the form $a_p x^p$ is continuous everywhere.

5.11 Theorem. If $f(x)$ is a polynomial and a and b are real numbers such that $f(a) > 0$ and $f(b) < 0$, then $f(x) = 0$ has a root between a and b.

This theorem depends on the fact that a polynomial is a continuous single-valued function. No proof is given here, but it is suggested that the student consider the geometric significance of the theorem.

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$$| a_0 x^n | - | a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n | > 0. \quad (5.16)$$

Now, for $x > 0$,

$$| a_0 x^n | - | a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1}x + a_n | \quad (5.17)$$

$$\geqq | a_0 | x^n - | a_1 | x^{n-1} - | a_2 | x^{n-2} \cdots - | a_{n-1} | x - | a_n | \quad (5.18)$$

$$\geqq | a_0 | x^n - | a_k | x^{n-1} - | a_k | x^{n-2} \cdots - | a_k | x - | a_k | \quad (5.19)$$

nomial when written in the form of (5.27). Substituting $x = 5$ in (5.27) we have

$$f(5) = [(2 \cdot 5 + 3)5 - 2]5 + 5. \tag{5.28}$$

The work should be arranged as follows: Write the coefficients of the polynomial on one line. Indicate the number being substituted to the right, for convenience,

$$2 + 3 - 2 + 5 \quad \underline{/+5}.$$

Leaving space for a row of figures, draw a line below the coefficients and copy the first coefficient beneath the line. We have

$$\begin{array}{l} 2 + 3 - 3 + 5 \quad \underline{/+5} \\ \\ \hline 2 \end{array}$$

Now $f(5)$ is evaluated, see equation (5.28), by first taking 2 times 5 equals 10, then 10 plus 3 equals 13. These steps are written in and we have

$$\begin{array}{llll} 2\,+ & 3 - & 2 + 5 & \underline{/+5} \\ & 10 & & \\ \hline 2\,+ & 13 & & \end{array}$$

This is continued, see equation (5.28), 13 times 5 equals 65, 65 minus 2 equals 63 and we have

$$\begin{array}{lllll} 2\,+ & 3\,- & 2 + 5 & & \underline{/+5} \\ & 10\,+ & 65 & & \\ \hline 2\,+ & 13\,+ & 63 & & \end{array}$$

And, finally, 63 times 5 equals 315, 315 plus 5 equals 320

$$\begin{array}{lllll} 2\,+ & 3\,- & 2\,+ & 5 & \underline{/+5} \\ & 10\,+ & 65\,+ & 315 & \\ \hline 2\,+ & 13\,+ & 63\,+ & 320 & \end{array}$$

and $320 = f(5)$.

If some power of x is missing in the polynomial its coefficient is written in as zero.

Example. Find $f(2)$ for $f(x) = x^4 + 3x^2 - 2x + 5$

$$\begin{array}{llllll} 1\,+ & 0\,+ & 3\,- & 2\,+ & 5 & \underline{/+2} \\ & 2\,+ & 4\,+ & 14\,+ & 24 & \\ \hline 1\,+ & 2\,+ & 7\,+ & 12\,+ & 29 & \end{array}$$

Therefore, $f(2) = 29$.

Example 1. Transform the equation $x^2 + 3x + 2 = 0$ into an equation whose roots are the reciprocals of the roots of the original equation. Let $x = y^{-1}$ and we have

$$\frac{1}{y^2} + \frac{3}{y} + 2 = 0.$$

Multiply by y^2 and this becomes

$$2y^2 + 3y + 1 = 0.$$

Example 2. Transform the equation $x^3 + 2x + 5 = 0$ into an equation whose roots are the reciprocals of the corresponding roots of the given equation. Let $x = y^{-1}$.

$$\frac{1}{y^3} + \frac{2}{y} + 5 = 0.$$

Multiply by y^3

$$5y^3 + 2y^2 + 1 = 0$$

is the desired equation.

5.22 Theorem. To transform an equation of degree n into an equation each of whose roots is less by k than the roots of the given equation, we proceed as follows: Divide the polynomial by $(x - k)$ and denote the remainder by R_n; divide the quotient by $(x - k)$ and denote the new remainder by R_{n-1}; continue this process until n remainders are found. Then

$$a_0y^n + R_1y^{n-1} + R_2y^{n-2} + \cdots + R_{n-1}y + R_n = 0 \tag{5.39}$$

is the required equation.

Assuming that equation (5.39) is the desired equation we can show that the coefficients can be obtained as described above. Since the roots of the y-equation are each k less than the roots of the x-equation, substituting $y = x - k$ in the y-equation will give the x-equation. Therefore, the equation

$$\begin{aligned} a_0(x - k)^n + R_1(x - k)^{n-1} + R_2(x - k)^{n-2} \\ + \cdots + R_{n-1}(x - k) + R_n = 0 \end{aligned} \tag{5.40}$$

is the same as the given equation. When the given equation is written in this form it is evident that R_n is the remainder after dividing by $x - k$, and R_{n-1} is the remainder after dividing the quotient by $x - k$, etc.

This transformation is simplified if synthetic division is used. As an example: Given the equation $x^3 + 2x^2 - x + 1 = 0$; required the equation whose roots are 2 less than the corresponding roots of the given equa-

tion. We set up the tabulation as directed for synthetic division

$$\begin{array}{rrrr|l} 1 & +2 & -1 & +1 & \underline{/+2} \\ & 2 & +8 & +14 & \\ \hline 1 & +4 & +7 & +15 & \\ & 2 & +12 & & \\ \hline 1 & +6 & +19 & & \\ & 2 & & & \\ \hline 1 & +8 & & & \end{array}$$

The first remainder is 15. The first quotient is $x^2 + 4x + 7$. When it is divided by $x - 2$ the next remainder is 19 and the quotient $x + 6$. When $x + 6$ is divided by $x - 2$ the remainder is 8. The transformed equation is, therefore, $y^3 + 8y^2 + 19y + 15 = 0$.

5.23 Theorem. Given a polynomial of degree n

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \tag{5.41}$$

We can eliminate the term containing x^{n-1} by substituting $x = y - a_1/n$.

When this substitution is made we have

$$\left(y - \frac{a_1}{n}\right)^n + a_1\left(y - \frac{a_1}{n}\right)^{n-1} + \cdots + a_{n-1}\left(y - \frac{a_1}{n}\right) + a_n = 0. \tag{5.42}$$

When this is multiplied out only the first two terms will contain y^{n-1}, the coefficient of y^{n-1} in the first term is $-a_1$, and the coefficient of y^{n-1} in the second term is $+a_1$. These cancel and therefore there is no term in the resulting equation containing y^{n-1}. This process is the first step in solving equations by certain processes, for example, the trigonometric solution of the cubic equation section 5.29.

5.24 Bounds to the Real Roots of an Equation. It was shown in section 5.13 that for $x > 1 + \left|\frac{na_k}{a_0}\right|$ the polynomial $f(x)$ has the same sign as a_0, where a_k is the coefficient having the largest absolute value of $a_1, a_2, \cdots a_n$. This means that for $x > 1 + \left|\frac{na_k}{a_0}\right|$, $f(x) \neq 0$; therefore the equation $f(x) = 0$ cannot have a real root larger than $1 + \left|\frac{na_k}{a_0}\right|$. Therefore, $1 + \left|\frac{na_k}{a_0}\right|$ is an upper bound to the real roots of $f(x) = 0$. This upper bound is usually rather large. The smaller the upper bound

that we can find for the real roots of an equation the closer we come to finding the largest real root.

In order to obtain the real roots of an equation by Horner's method and by Newton's method (these methods are described in the next chapter), we must first find an approximate value for the root. A knowledge of the bounds of the roots may be a great help.

5.25 An Upper Bound. A smaller upper bound than the one specified in the preceding section can be found by the following device. Given the equation of degree n

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \tag{5.43}$$

Divide through by a_0

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_{n-1}}{a_0}\,x + \frac{a_n}{a_0} = 0. \tag{5.44}$$

Place parentheses around the terms as follows:

$$\left(\left(\cdots\left(x + \frac{a_1}{a_0}\right)x + \frac{a_2}{a_0}\right)x + \frac{a_3}{a_0}\right)x + \cdots\Bigg)x + \frac{a_{n-1}}{a_0}\Bigg)x + \frac{a_n}{a_0}\right) = 0. \tag{5.45}$$

The notation (($\cdots$ (indicates that the n sets of parentheses all start at the same place. We might indicate this bv the symbol $(^n$. If we make $(x + a_1/a_0) > 1$ and $x > 1$, then

$$\left(x + \frac{a_1}{a_0}\right)x > x,$$

$$\left(x + \frac{a_1}{a_0}\right)x + \frac{a_2}{a_0} > x + \frac{a_2}{a_0}.$$

The above will be greater than one if we make $(x + a_1/a_0) > 1$, $x > 1$, and $(x + a_2/a_0) > 1$, and therefore

$$\left(\left(x + \frac{a_1}{a_0}\right)x + \frac{a_2}{a_0}\right)x + \frac{a_3}{a_0} > \left(x + \frac{a_2}{a_0}\right)x + \frac{a_3}{a_0} > x + \frac{a_3}{a_0}.$$

Finally, if we make $(x + a_1/a_0) > 1$, $x > 1$, $(x + a_2/a_0) > 1, \cdots (x + a_n/a_0) > 1$, then $f(x) > 1$. These conditions are satisfied if x is larger then the largest of

$$1, \quad 1 - \frac{a_1}{a_0}, \quad 1 - \frac{a_2}{a_0}, \quad \cdots 1 - \frac{a_n}{a_0}.$$

If x is greater than the largest number above $f(x) > 1$, the largest of 1, $1 - a_1/a_0, \cdots 1 - a_n/a_0$ is an upper bound to the real roots of $f(x) = 0$.

To apply this test we need only consider those coefficients that differ in

sign with a_0. This upper bound can be described as follows so that it can be compared with the upper bound found in section 5.13. $1 + \left|\frac{a_k}{a_0}\right|$ is an upper bound where a_k is the coefficient having greatest absolute value of all those that differ in sign with a_0. If no coefficient differs in sign from a_0 there can be no positive roots and zero is an upper bound to the real roots.

Example. Consider the equation

$$2x^4 + 8x^3 - 7x^2 + 16x - 3 = 0.$$

An upper bound is

$$1 + \tfrac{7}{2} = 4.5.$$

The formula for an upper bound derived in section 5.13 would give in this case $1 + (4 \cdot 16)/2 = 33$. Both these results are upper bounds but 4.5 is much more useful than 33.

5.26 A Lower Bound to the Positive Roots. If the given equation is transformed into an equation whose roots are the reciprocals of the roots of the original equation, the upper bound to the roots of the transformed equation will be greater than the reciprocals of the roots of the given equation. Therefore its reciprocal will be less than any of the positive roots of the given equation.

Given the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \tag{5.46}$$

Transform this equation by the method of section 5.21

$$a_ny^n + a_{n-1}y^{n-1} + \cdots + a_1y + a_0 = 0. \tag{5.47}$$

An upper bound to the roots of equation (5.47) is $1 + \left|\frac{a_k}{a_n}\right|$ where a_k is the coefficient having the largest absolute value of all those that differ in sign with a_n. Therefore a lower bound to the positive roots of equation (5.46) is, where a_k is as just defined,

$$\frac{1}{1 + \left|\dfrac{a_k}{a_n}\right|} = \frac{|\,a_n\,|}{|\,a_n\,| + |\,a_k\,|}.$$

Example. A lower bound to the real positive roots of the equation

$$2x^4 + 8x^3 + 16x^2 - 47x - 3 = 0$$

is

$$\frac{3}{3 + 16} = \frac{3}{19}.$$

5.27 A Lower Bound to the Real Roots. If the given equation is transformed into a new equation each of whose roots is the negative of the corresponding root of the given equation, an upper bound to the roots of the transformed equation will be greater than the negatives of the roots of the given equation; therefore, its negative will be less than the roots of the given equation. Given the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0. \tag{5.48}$$

Transform this by the method of section 5.20.

$$a_0y^n - a_1y^{n-1} + a_2y^{n-2} - \cdots (-1)^{n-1}a_{n-1}y + (-1)^n a_n = 0. \tag{5.49}$$

An upper bound to the roots of equation (5.49) is $1 + \left|\dfrac{a_k}{a_0}\right|$ where $a_k(-1)^k$ differs in sign with a_0 and is numerically the greatest coefficient that satisfies this requirement as to sign. Therefore a lower bound to the real roots of the given equation is $-1 - \left|\dfrac{a_k}{a_0}\right|$ where a_k is as defined above.

Example. In the equation

$$2x^4 + 8x^3 - 7x^2 + 16x - 3 = 0$$

the coefficients $(-1)^k a_k$ are

$$2, \qquad -8, \qquad -7, \qquad -16, \qquad -3.$$

A lower bound to the real roots of the given equation is

$$-1 - \tfrac{16}{2} = -9.$$

5.28 An Upper Bound to the Negative Roots. By combining the steps in the preceding sections we find an upper bound to the negative roots of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \tag{5.50}$$

to be $-\dfrac{|a_n|}{|a_n| + |a_k|}$, where $a_k(-1)^k$ differs in sign with $a_n(-1)^n$ and is numerically the greatest coefficient that satisfies this requirement as to sign.

Example. To find an upper bound to the negative roots of

$$2x^4 + 8x^3 - 7x^2 + 16x - 3 = 0$$

rewrite the coefficients as $(-1)^k a_k$ for convenience

$$2, \qquad -8, \qquad -7, \qquad -16, \qquad -3.$$

An upper bound to the negative roots is $-3/(3 + 2) = -0.6$.

5.29 Trigonometric Solution of the Cubic Equation. The following method can be used to solve any equation of the third degree. If the given equation contains a term involving x^2 it can be eliminated by the method of section 5.23 so that we can start with the equation in the form

$$x^3 + ax + b = 0. \tag{5.51}$$

We wish to transform our equation to the trigonometric identity

$$\cos^3\theta - \tfrac{3}{4}\cos\theta - \tfrac{1}{4}\cos 3\theta \equiv 0. \tag{5.52}$$

To do this substitute $x = my$ in equation (5.51)

$$m^3y^3 + amy + b = 0. \tag{5.53}$$

Dividing equation (5.53) by m^3 we have

$$y^3 + \frac{a}{m^2}y + \frac{b}{m^3} = 0. \tag{5.54}$$

Comparing equations (5.54) and (5.52) we can have $y = \cos\theta$, provided that $\dfrac{a}{m^2} = -\tfrac{3}{4}$ and $\dfrac{b}{m^3} = -\tfrac{1}{4}\cos 3\theta$. That is,

$$m = 2\sqrt{-\frac{a}{3}}, \tag{5.55}$$

$$\cos 3\theta = \frac{3b}{am}. \tag{5.56}$$

If θ_1 satisfies equation (5.56), $\theta_1 + 2\pi/3$ and $\theta_1 + 4\pi/3$ also satisfy equation (5.56). Therefore the three roots of equation (5.54) are $\cos\theta_1$, $\cos(\theta_1 + 2\pi/3)$, $\cos(\theta_1 + 4\pi/3)$, and the roots of the given equation are

$$m\cos\theta_1, \quad m\cos\left(\theta_1 + \frac{2\pi}{3}\right), \quad m\cos\left(\theta_1 + \frac{4\pi}{3}\right).$$

The three cases that may arise are illustrated in the three examples that follow.

Example 1. Required the roots of the equation

$$x^3 - 2x + 1 = 0$$
$$a = -2,\ b = 1$$
$$m = 2\sqrt{\tfrac{2}{3}} = 1.633$$
$$\cos 3\theta = \frac{3}{-2(1.633)} = -0.9185$$
$$3\theta = 156°\,43'$$

$$\begin{aligned} \theta &= 52°\,14' & \cos\theta &= 0.612 \\ \theta + 120° &= 172°\,14' & \cos 172°\,14' &= -0.991 \\ \theta + 240° &= 292°\,14' & \cos 292°\,14' &= 0.378 \end{aligned}$$

The required roots are

$$\begin{aligned}(1.633)(0.612) &= 1\\(1.633)(-0.991) &= -1.62\\(1.633)(0.378) &= 0.617\end{aligned}$$

Note: we could have taken 3θ in the third quadrant instead of in the second quadrant. The result would be the same since we use only $\cos\theta$, $\cos(\theta + 120°)$, and $\cos(\theta + 240°)$.

Example 2. Required the roots of the equation

$$x^3 + 2x + 1 = 0$$

$$a = 2,\ b = 1$$

$$m = 2\sqrt{\frac{-2}{3}} = i1.633$$

$$\cos 3\theta = \frac{3}{2(i1.633)} = -i0.9185$$

Now, $\cos 3\theta = \sin(3\theta + \pi/2) = -i \sinh i(3\theta + \pi/2)$. Therefore,

$$\sinh i\left(3\theta + \frac{\pi}{2}\right) = 0.9185$$

From tables of hyperbolic functions we find

$$i\left(3\theta + \frac{\pi}{2}\right) = 0.823$$

$$3\theta = -\frac{\pi}{2} - i0.823$$

$$\theta = -\frac{\pi}{6} - i0.274$$

$$\theta + \frac{2\pi}{3} = \frac{\pi}{2} - i0.274$$

$$\theta + \frac{4\pi}{3} = \frac{7\pi}{6} - i0.274$$

$$\begin{aligned}\cos\theta &= \cos\frac{\pi}{6}\cosh 0.274 - i\sin\frac{\pi}{6}\sinh 0.274\\&= (0.866)(1.038) - i(0.5)(0.277) = 0.898 - i0.139\\\cos\left(\theta + \frac{2\pi}{3}\right) &= \cos\frac{\pi}{2}\cosh 0.274 + i\sin\frac{\pi}{2}\sinh 0.274\\&= 0 + i0.277\\\cos\left(\theta + \frac{4\pi}{3}\right) &= \cos\frac{7\pi}{6}\cosh 0.274 + i\sin\frac{7\pi}{6}\sinh 0.274\\&= (-0.866)(1.038) + i(-0.5)(0.277) = -0.898 - i0.139\end{aligned}$$

The required roots are

$$\begin{aligned}i(1.633)(0.898 - i0.139) &= 0.226 + i1.466\\i(1.633)(i0.277) &= -0.452\\i(1.633)(-0.898 - i0.139) &= 0.226 - i1.466\end{aligned}$$

Example 3. Required the roots of the equation

$$x^3 - x + 2 = 0$$
$$a = -1, b = 2$$
$$m = 2\sqrt{\tfrac{1}{3}} = 1.155$$
$$\cos 3\theta = \frac{6}{-1.155} = -5.196$$

Now,

$$\cos 3\theta = -\cos(3\theta - \pi) = -\cosh i(3\theta - \pi) = -5.196.$$

From tables of hyperbolic functions we find for $\cosh i(3\theta - \pi) = 5.196$ that $i(3\theta - \pi) = 2.332$. Therefore,

$$3\theta - \pi = -i2.332$$
$$3\theta = \pi - i2.332$$
$$\theta = \frac{\pi}{3} - i0.777$$
$$\theta + \frac{2\pi}{3} = \pi - i0.777$$
$$\theta + \frac{4\pi}{3} = \frac{5\pi}{3} - i0.777$$

$$\begin{aligned}
\cos\theta &= \cos\frac{\pi}{3}\cosh 0.777 + i\sin\frac{\pi}{3}\sinh 0.777 \\
&= (0.5)(1.317) + i(0.866)(0.858) = 0.659 + i0.743 \\
\cos\left(\theta + \frac{2\pi}{3}\right) &= \cos\pi\cosh 0.777 + i\sin\pi\sinh 0.777 \\
&= -1.317 \\
\cos\left(\theta + \frac{4\pi}{3}\right) &= \cos\frac{5\pi}{3}\cosh 0.777 + i\sin\frac{5\pi}{3}\sinh 0.777 \\
&= (0.5)(1.317) - i(0.866)(0.858) = 0.659 - i0.743
\end{aligned}$$

The required roots are

$$\begin{aligned}
1.155(0.659 + i0.743) &= 0.760 + i0.858 \\
1.155(-1.317) &= -1.520 \\
1.155(0.659 - i0.743) &= 0.760 - i0.858
\end{aligned}$$

PROBLEMS ON CHAPTER 5

1. Given $f(x) = x^3 + 3x^2 - x - 15$. Find $f(2)$ by substitution and by dividing by $x - 2$.
2. Given $f(x) = x^4 + 3x^2 - 2x - 10$. Find $f(3)$ by substitution and by using the remainder theorem.
3. Given $f(x) = x^4 - 4x^3 + 3x - 3$. Find $f(5)$ by substitution and by using the remainder theorem.
4. Show that if $f(x)$ and $g(x)$ are both continuous at $x = a$ their product is continuous at $x = a$.

5. Show that if $f(x)$ and $g(x)$ are continuous at $x = a$ their sum is continuous at $x = a$.
6. Show that $f(x) = x$ is continuous everywhere.
7. Show that $f(x) = x^p$, where p is a non-negative integer, is continuous everywhere.
8. Show that a polynomial is continuous everywhere.
9. Draw some graphs that illustrate the truth of the theorem in section 5.11.
10. Use synthetic division on problem 1.
11. Use synthetic division on problem 2.
12. Use synthetic division on problem 3.
13. Transform the equation $2x^3 - 3x^2 + 2x - 10 = 0$ into an equation whose roots are three times the roots of the given equation.
14. Transform the equation $5x^4 - x^2 + 2x - 7 = 0$ into an equation whose roots are one third the roots of the given equation.
15. Transform the equation $5x^5 - 7x^2 - x + 10 = 0$ into an equation whose roots are two times the roots of the given equation.
16. Transform the equation in problem 13 into an equation whose roots are the negatives of the roots of the given equation.
17. Repeat problem 16 for the equation in problem 14.
18. Repeat problem 16 for the equation in problem 15.
19. Transform the equation in problem 13 into an equation whose roots are the reciprocals of the roots of the given equation.
20. Repeat problem 19 for the equation in problem 14.
21. Repeat problem 19 for the equation in problem 15.
22. Transform the equation in problem 13 into a new equation whose roots are twelve more than the roots of the given equation.
23. Transform the equation in problem 14 into a new equation whose roots are two less than the roots of the given equation.
24. Transform the equation in problem 15 into a new equation whose roots are three less than the roots of the given equation.
25. Eliminate the x^2 term in the equation $x^3 - 3x^2 + x - 10 = 0$.
26. Eliminate the x^3 term in $x^4 + x^3 + x - 2 = 0$.
27. Eliminate the x^3 term in $2x^4 + x^3 - 3 = 0$.
28. Find an upper bound to the real roots of the equation in problem 13.
29. Find an upper bound to the real roots of the equation in problem 14.
30. Find an upper bound to the real roots of the equation in problem 15.
31. Find a lower bound to the positive real roots of the equation in problem 13.
32. Find a lower bound to the positive real roots of the equation in problem 14.
33. Find a lower bound to the positive real roots of the equation in problem 15.
34. Find a lower bound to the real roots of the equation in problem 13.
35. Find a lower bound to the real roots of the equation in problem 14.
36. Find a lower bound to the real roots of the equation in problem 15.
37. Prove that the formula given in section 5.28 gives an upper bound to the negative real roots of an equation.
38. Find an upper bound to the negative roots of the equation in problem 13.
39. Find an upper bound to the negative roots of the equation in problem 14.
40. Find an upper bound to the negative roots of the equation in problem 15.
41. Bound the real roots of the equation in problem 25.
42. Bound the real roots of the equation in problem 26.
43. Bound the real roots of the equation in problem 27.

44. Find the roots of $x^3 - 8x - 2 = 0$.
45. Find the roots of $x^3 - 5x + 3 = 0$.
46. Find the roots of $x^3 - 2x + 12 = 0$.
47. Find the roots of $x^3 - 3x - 10 = 0$.
48. Find the roots of $x^3 + 2x - 5 = 0$.
49. Find the roots of $x^3 + 3x + 15 = 0$.
50. Find the roots of $x^3 + 2x - 1 = 0$.
51. Find the roots of the equation in problem 13.
52. Find the roots of the equation in problem 25.

CHAPTER 6

APPROXIMATE SOLUTIONS OF ALGEBRAIC EQUATIONS

6.1 Derivatives. Since this book is intended for students that have studied the calculus it will be assumed that the process of differentiation is understood and that the student can differentiate the polynomial

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n \tag{6.1}$$

and obtain

$$nx^{n-1} + (n-1)a_1x^{n-2} + (n-2)a_2x^{n-3} + \cdots + 2xa_{n-2} + a_{n-1}. \tag{6.2}$$

6.2 Multiple Roots. The factor theorem states that, if r is a root of the polynomial equation $f(x) = 0$, $f(x)$ is divisible by $x - r$ or $x - r$ is a factor of $f(x)$. It may happen that $(x - r)^2$ is a factor of $f(x)$, and in this case the polynomial equation $f(x) = 0$ of degree n will have less than n distinct roots. If $x - r$ is a factor of $f(x)$ and $(x - r)^2$ is not a factor, then r is a simple root of $f(x) = 0$. If $(x - r)^2$ is a factor of $f(x)$ and $(x - r)^3$ is not a factor, then r is a double root of $f(x) = 0$. In general we define a multiple root as follows: r is an m-fold root of $f(x) = 0$ if $(x - r)^m$ is a factor of $f(x)$ and $(x - r)^{m+1}$ is not a factor.

6.3 If we write the polynomial $f(x)$ in the form of Taylor's theorem near $x = b$, we have

$$f(x) = f(b) + (x - b)f'(b) + \frac{(x-b)^2}{2}f''(b) + \frac{(x-b)^3}{3!}f'''(b) + \cdots + \frac{(x-b)^n}{n!}f^{(n)}(b). \tag{6.3}$$

If b is a simple root of $f(x) = 0$, $f(x)$ is divisible by $x - b$ and not by $(x - b)^2$. Therefore $f(b) = 0$ and $f'(b) \neq 0$. If b is a double root of $f(x) = 0$, $f(x)$ is divisible by $(x - b)^2$ and not by $(x - b)^3$. Therefore $f(b) = f'(b) = 0$ and $f''(b) \neq 0$. In general, if b is an m-fold root of $f(x) = 0$, $f(x)$ is divisible by $(x - b)^m$ and not by $(x - b)^{m+1}$. Therefore $f(b) = f'(b) = f''(b) = \cdots = f^{(m-1)}(b) = 0$ and $f^{(m)}(b) \neq 0$. This establishes the following:

Theorem. A necessary and sufficient condition that r be an m-fold root of $f(x) = 0$ is that $f(r) = f'(r) = f''(r) = \cdots = f^{(m-1)}(r) = 0$ and $f^{(m)}(r) \neq 0$.

6.4 The theorem just established may be stated: A necessary and sufficient condition that r be an $(m-1)$-fold root of $g(x) = 0$ is that $g(r) = g'(r) = g''(r) = \cdots = g^{(m-2)}(r) = 0$ and $g^{(m-1)}(r) \neq 0$.

Let $g(x) = f'(x)$, $g'(x) = f''(x)$, etc. We have:

Theorem. A necessary and sufficient condition that r be an $(m-1)$-fold root of $f'(x) = 0$ is that $f'(r) = f''(r) = \cdots f^{(m-1)}(r) = 0$ and $f^{(m)}(r) \neq 0$. This and the preceding theorem give:

Theorem. A necessary and sufficient condition that a root of $f(x) = 0$ be an m-fold root is that it be an $(m-1)$-fold root of $f'(x) = 0$.

6.5 If this process is repeated on successive derivatives we obtain a result which can be summarized in the following table.

TABLE VI–1

Equation			*Type of Root*		
$f(x) = 0$	simple	double	triple	4-fold	5-fold
$f'(x) = 0$	no	simple	double	triple	4-fold
$f''(x) = 0$	?	no	simple	double	triple
$f'''(x) = 0$	?	?	no	simple	double
$f^{\text{IV}}(x) = 0$	?	?	?	no	simple

The question marks indicate ambiguities. A simple root of $f(x) = 0$ can be a root of any of the equations except $f'(x) = 0$. A double root of $f(x) = 0$ is a root of $f'(x) = 0$, is not a root of $f''(x) = 0$, and can be a root of the other equations.

6.6 Highest Common Factor. The highest common factor, abbreviated H.C.F., of two polynomials in x is the polynomial of highest degree containing x which is a factor of the two given polynomials. The H.C.F. of two polynomials can be found by a process almost identical with that used to find the greatest common divisor of two integers. Let A and B be two polynomials and assume the degree of A is not less than the degree of B. Divide A by B and obtain a quotient Q_1 and a remainder R_1 which is of lower degree than B. If $R_1 = 0$, B is the H.C.F. Assume $R_1 \neq 0$ and divide B by R_1 and obtain a quotient Q_2 and a remainder R_2 of lower degree than R_1; continue the process until the remainder is a constant (of zero degree).

$$\begin{aligned} A &= BQ_1 + R_1. \\ B &= R_1Q_2 + R_2. \\ R_1 &= R_2Q_3 + R_3. \\ R_2 &= R_3Q_4 + R_4. \end{aligned} \tag{6.4}$$

Assume R_4 is a constant. If $R_4 = 0$, R_3 is a factor of R_2, and since R_3 is a factor of R_2 it is a factor of R_1, and since R_3 is a factor of R_2 and R_1 it

is a factor of B. Since R_3 is a factor of R_1 and B it is a factor of A. This shows that R_3 is a common factor of A and B. Now the H.C.F. of A and B is a factor of R_1, and since it is a factor of B and R_1 it is a factor of R_2, and since the H.C.F. of A and B is a factor of R_1 and R_2 it is a factor of R_3. Since R_3 is a common factor of A and B it must the H.C.F. If $R_4 \neq 0$ there is no common factor of A and B containing x.

6.7 Let $g_2(x)$ be the H.C.F. of $f(x)$ and $f'(x)$. Then every root of $g_2(x) = 0$ is a multiple root of $f(x) = 0$, and an m-fold root of $f(x) = 0$ is an $(m - 1)$-fold root of $g_2(x) = 0$ just as it is of $f'(x) = 0$. Let $g_3(x)$ be the H.C.F. of $g_2(x)$ and $g_2'(x)$. Every root of $g_3(x) = 0$ is a triple or higher order root of $f(x) = 0$, and triple roots of $f(x) = 0$ are simple roots of $g_3(x) = 0$. This process can be used to define additional polynomials whose properties are summarized in Table VI–2.

TABLE VI–2

Equation			*Type of Root*		
$f(x) = 0$	simple	double	triple	4-fold	5-fold
$g_2(x) = 0$	no	simple	double	triple	4-fold
$g_3(x) = 0$	no	no	simple	double	triple
$g_4(x) = 0$	no	no	no	simple	double
$g_5(x) = 0$	no	no	no	no	simple

6.8 Let $P_1 = (x - r_1)(x - r_2) \cdots$, where r_1, r_2, etc. are all the simple roots of $f(x) = 0$. Let $P_2 = (x - s_1)(x - s_2) \cdots$, where s_1, s_2, etc. are all the double roots of $f(x) = 0$. Define additional polynomials in the same way. The properties of these polynomials are summarized in Table VI–3.

TABLE VI–3

Equation			*Type of Root*		
$f(x) = 0$	simple	double	triple	4-fold	5-fold
$P_1 = 0$	simple	no	no	no	no
$P_2 = 0$	no	simple	no	no	no
$P_3 = 0$	no	no	simple	no	no
$P_4 = 0$	no	no	no	simple	no

6.9 These two sets of polynomials are related in the following way:

$$
\begin{aligned}
f(x) &= P_1P_2^2P_3^3P_4^4P_5^5 \cdots \\
g_2(x) &= P_2P_3^2P_4^3P_5^4 \cdots \\
g_3(x) &= P_3P_4^2P_5^3P_6^4 \cdots \\
\\
g_m(x) &= P_mP_{m+1}^2P_{m+2}^3P_{m+3}^4 \cdots \\
g_{m+1}(x) &= P_{m+1}P_{m+2}^2P_{m+3}^3P_{m+4}^4 \cdots \\
g_{m+2}(x) &= P_{m+2}P_{m+3}^2P_{m+4}^3P_{m+5}^4 \cdots
\end{aligned}
\tag{6.5}
$$

$$\begin{aligned} g_m(x)g_{m+2}(x) &= P_m P_{m+1}^2 P_{m+2}^4 P_{m+3}^6 P_{m+4}^8 \cdots \\ &= P_m[P_{m+1}P_{m+2}^2 P_{m+3}^3 P_{m+4}^4 \cdots]^2 \\ &= P_m g_{m+1}^2(x) \end{aligned}$$

Therefore

$$P_m = \frac{g_m(x)g_{m+2}(x)}{g_{m+1}^2(x)}. \qquad P_1 = \frac{f(x)g_3(x)}{g_2^2(x)}. \tag{6.6}$$

Note that $g_1(x) = f(x)$

$$P_2 = \frac{g_2(x)g_4(x)}{g_3^2(x)}.$$

If a term is missing in these formulas, replace it by unity. If there are only simple and double roots, then

$$P_1 = \frac{f(x)}{g_2^2(x)}.$$

$$P_2 = g_2(x).$$

The problem of finding the roots of the equation $f(x) = 0$ is replaced by the problem of finding the roots of the equations $P_1 = 0$, $P_2 = 0$, etc. If P_1 is the same as $f(x)$, $f(x) = 0$ has no multiple roots; but, if P_1 is different from $f(x)$, the equation $f(x) = 0$ has multiple roots, and the chances are that some of the $P_m = 0$ equations are of the first or second degree, which is a great simplification. We shall assume that multiple roots are separated this way and our equations now have only simple roots.

6.10 Symmetric Roots. If $+r$ and $-r$ are roots of the equation $f(x) = 0$, r is a symmetric root of $f(x) = 0$. To test for symmetric roots we proceed as follows: Let $R_1(x)$ equal the polynomial containing all terms of $f(x)$ of odd degree. Let $R_2(x)$ equal all terms of $f(x)$ of even degree including the constant term. Then

$$R_1(x) + R_2(x) \equiv f(x). \tag{6.7}$$

If r is a root of $f(x) = 0$, we have

$$R_1(r) + R_2(r) = f(r) = 0. \tag{6.8}$$

And if $-r$ is a root of $f(x) = 0$, we have

$$R_1(-r) + R_2(-r) = f(-r) = 0. \tag{6.9}$$

Since $R_1(x)$ contains only terms of odd degree

$$R_1(-x) = -R_1(x). \tag{6.10}$$

And since $R_2(x)$ contains no terms of odd degree

$$R_2(-x) = R_2(x). \tag{6.11}$$

Equations (6.10) and (6.11) enable us to write equation (6.9)

$$-R_1(r) + R_2(r) = 0. \tag{6.12}$$

If r is a symmetric root of $f(x) = 0$, equations (6.8) and (6.12) are simultaneous, and their sum gives

$$R_2(r) = 0 \tag{6.13}$$

and their difference gives

$$R_1(r) = 0. \tag{6.14}$$

Theorem. A necessary and sufficient condition that $+r$ and $-r$ be roots of $f(x) = 0$ is that they be roots of the equation formed by taking the terms of odd degree and the equation formed by taking terms of even degree and the constant term of $f(x)$.

6.11 If $F(x)$ is the H.C.F. of $R_1(x)$ and $R_2(x)$, where $R_1(x)$ and $R_2(x)$ are defined in the preceding paragraph, $F(x) = 0$ contains all the pairs of symmetric roots of $f(x) = 0$, and $f(x)/F(x) = 0$ contains all the roots of $f(x) = 0$ except the symmetric roots. $F(x) = 0$ will be an equation every term of which is of even degree. Replace x^2 by y and let $G(y) = G(x^2) = F(x)$. $G(y) = 0$ may have symmetric roots as in the following example:

$$F(x) = (x-1)(x+1)(x-i)(x+i) = (x^2-1)(x^2+1),$$
$$G(y) = (y-1)(y+1).$$

The equation $G(y) = 0$ can be tested for symmetric roots in the same way that $f(x) = 0$ was tested. We obtain an equation $F_2(y) = 0$ which contains all symmetric roots of $G(y) = 0$. $G(y)/F_2(y) = 0$ contains all the roots of $G(y) = 0$ except the symmetric roots. Replace y^2 in $F_2(y) = 0$ by z and obtain $G_2(z) = 0$. This process is straightforward. We shall find it convenient to remove multiple roots and symmetric roots before applying Graeffe's method of approximating the roots of an equation.

6.12 Horner's Method. The real roots of a polynomial equation can be found to any desired number of decimal places by a process known as Horner's method. This process can best be explained by giving an example. If we require the root of the equation $f(x) = x^3 + 12x^2 + x - 100 = 0$ which lies between 2 and 3, the method is as follows:

(a) Transform the given equation into an equation whose roots are two

less than the roots of the given equation.

1	12	1	−100	/+2
	2	28	58	
1	14	29	−42	
	2	32		
1	16	61		
	2			
1	18			

The transformed equation is $f_1(x_1) = x_1^3 + 18x_1^2 + 61x_1 - 42 = 0$. The corresponding root of this equation is between 0 and 1.0.

(b) We can determine the sign of $f_1(0.1)$, $f_1(0.2)$, $f_1(0.3)$, etc., until we find two of opposite sign. We might try eight values before finding the change in sign. To reduce the amount of this trial and error work we perform the following trial division. Neglect the terms containing higher powers of the unknown. The equation is then $61x_1 - 42 = 0$. Solving this for x_1 we have $x_1 = \frac{42}{61} = 0.6+$. Determine $f_1(0.6)$

1	18	61	−42	/0.6
	0.6	11.16	43.296	
1	18.6	72.16	1.296	

Since $f_1(0.6) = 1.296 > 0$, and $f_1(0) = -42 < 0$, the root is between 0 and 0.6. Determine $f_1(0.5)$

1	18	61	−42	/0.5
	0.5	9.25	35.125	
1	18.5	70.25	−6.875	

Since $f_1(0.5) = -6.875 < 0$ and $f_1(0.6) > 0$, the root lies between 0.5 and 0.6 and the desired root of the original equation lies between 2.5 and 2.6.

(c) Transform the last equation into a new equation each of whose roots is 0.5 less than the corresponding root of the equation in x_1. The work of determining $f_1(0.5)$ is part of this transformation and is repeated below to identify the steps in the solution; in practice this would not be repeated.

1	18	61	−42	/0.5
	0.5	9.25	35.125	
1	18.5	70.25	−6.875	
	0.5	9.5		
1	19.0	79.75		
	0.5			
1	19.5			

The transformed equation is $f_2(x_2) = x_2^3 + 19.5x_2^2 + 79.75x_2 - 6.875 = 0$. The corresponding root of this equation is between 0 and 0.1.

(d) The trial division is $x_2 = 6.875/79.75$. Since we want only one place it is simpler to take for the trial division $x_2 = 6.8/80 = 0.08+$. Therefore determine $f_2(0.08)$

1	19.5	79.75	−6.875	/0.08
	0.08	1.5664	6.505312	
1	19.58	81.3164	−0.369688	

$f_2(0.08) = -0.369688 < 0$. Now determine $f_2(0.09)$

1	19.5	79.75	−6.875	/0.09
	0.09	1.7631	7.336179	
1	19.59	81.5131	0.461179	

Since $f_2(0.09) = 0.461179 > 0$, the root is between 0.08 and 0.09 and the required root of the original equation is between 2.58 and 2.59.

(e) Transform the last equation into a new equation each of whose roots is 0.08 less than the corresponding root of the equation in x_2. Although we have already done part of this work above, the entire transformation is given below.

1	19.5	79.75	−6.875	/0.08
	0.08	1.5664	6.505312	
1	19.58	81.3164	−0.369688	
	0.08	1.5728		
1	19.66	82.8892		
	0.08			
1	19.74			

The new equation is

$$f_3(x_3) = x_3^3 + 19.74x_3^2 + 82.8892x_3 - 0.369688 = 0.$$

The corresponding root of this equation is between 0 and 0.01.

(f) Trial division gives $x_3 = 0.4/83 = 0.004+$. Determine $f_3(0.004)$

1	19.74	82.8892	−0.369688	/0.004
	0.004	0.078976	0.331872704	
1	19.744	82.968176	−0.037815296	

Now determine $f_3(0.005)$

1	19.74	82.8892	−0.369688	/0.005
	0.005	0.098725	0.414939625	
1	19.745	82.987925	0.045251625	

Since $f_3(0.004) = -0.037815296 < 0$ and $f_3(0.005) = 0.045251625 > 0$, the root is between 0.004 and 0.005 and the desired root of the original equation is between 2.584 and 2.585.

(g) Transform the last equation into a new equation each of whose roots is 0.004 less than the corresponding root of the equation in x_3.

1	19.74	82.8892	−0.369688	/0.004
	0.004	0.078976	0.331872704	
1	19.744	82.968176	−0.037815296	
	0.004	0.078992		
1	19.748	83.047168		
	0.004			
1	19.752			

The transformed equation is

$$f_4(x_4) = x_4^3 + 19.752x_4^2 + 83.047168x_4 - 0.037815296 = 0.$$

The root of this equation is between 0 and 0.001.

(h) Trial division gives $x_4 = 0.04/83 = 0.0004+$. Determine $f_4(0.0004)$.

1	19.752	83.047168	−0.037815296	/0.0004
	0.0004	0.00790096	0.033222027584	
1	19.7524	83.05506896	−0.004593268416	

Since $f_4(0.0004) = -0.004593268416 < 0$ determine $f_4(0.0005)$:

1	19.752	83.047168	−0.037815296	/0.0005
	0.0005	0.00987625	0.041528522125	
1	19.7525	83.05704425	0.003713226125	

Since $f_4(0.0004) < 0$ and $f_4(0.0005) = 0.003713226125 > 0$, the desired root is between 0.0004 and 0.0005, and the desired root of the original equation is between 2.5844 and 2.5845. We could continue this way getting one figure each time we transform the equation. However, when the desired root is so small, we can get several additional decimal places at once by adjusting the trial division as follows.

(i) Write the last equation as follows:

$$83.047168x_4 = 0.037815296 - x_4^3 - 19.752x_4^2.$$

Since the desired root is between 0.0004 and 0.0005

$$83.047168x_4 < 0.037815296 - 0.0004^3 - 19.752(0.0004)^2$$
$$83.047168x_4 > 0.037815296 - 0.0005^3 - 19.752(0.0005)^2$$

These inequalities are

$$83.047168x_4 < 0.037812135616$$
$$83.047168x_4 > 0.037810357875$$

To solve for limiting values of x_4 from these inequalities, we have two problems in long division which are placed side by side below.

$$\begin{array}{r}
0.0004553+ \\
83.047168\overline{)0.037812135616} \\
\underline{332188672} \\
459326841 \\
\underline{415235840} \\
440910016 \\
\underline{415235840} \\
256741760 \\
\underline{249141504} \\
7600256
\end{array}
\qquad
\begin{array}{r}
0.0004552+ \\
83.047168\overline{)0.037810357875} \\
\underline{332188672} \\
459149067 \\
\underline{415235840} \\
439132275 \\
\underline{415235840} \\
238964350 \\
\underline{166094336} \\
72870014
\end{array}$$

The desired value of x_4 is between 0.0004552 and 0.0004554, and the required root of the original equation is between 2.5844552 and 2.5844554.

(j) To obtain additional decimal places in the root we can repeat the steps in part (i) using the figures 0.0004552 and 0.0004554 in place of 0.0004 and 0.0005, respectively.

(k) The two inequalities in section (i) are solved for limiting values of x_4 by long division. The long division process is carried out in the two cases only as long as the figures in the quotients are the same. As soon as different figures arise there is no gain in going further. The work can be abbreviated if we use only the figures in the dividend that are alike and perform only one division process. In the example above we can divide 0.03781 by 83.047168 and expect to get the root correct to several places. Without much extra work we can take an extra place into account and divide 0.037812 by 83.047168. In this case we can then investigate what the result would have been if we had used 0.037810 for the dividend without repeating the entire process.

If we divide 0.037812 by 83.047168 we have the first figure in the quotient as 0.0004, and the first steps would appear

$$\begin{array}{r}
0.0004 \\
83.047168\overline{)0.037812} \\
\underline{332188672}
\end{array}$$

Now the figures in the dividend above 8672 in the division above were dropped because they are unreliable. It is false accuracy, therefore, to

carry the figures 8672 in the product, and we should drop some figures in the divisor. Knowing the first figure in the quotient, it is obvious that in the example above we should neglect the last four figures in the divisor. We place a dot over the 4 in the divisor as a reminder that the following figures are to be neglected. We do not neglect the last figures entirely since in the product we may have something to carry. In the illustration we have 4 (in the quotient) times 7 (to the right of the dot) equals 28, or 3 to carry. Four (in the quotient) times 4 (under the dot) equals 16; add the 3 we carried and get 19; the first step is completed as follows:

$$\begin{array}{r} 0.0004 \\ 83.0\dot{4}7168\overline{)0.037812} \\ \underline{33219} \\ 4593 \end{array}$$

We can now neglect an extra figure in the divisor and therefore we place a dot over the 0. The next figure in the quotient is 5. Therefore, 5 (in the quotient) times 4 (to the right of the last dot) is 20, or 2 to carry. Five (in the quotient) times 0 (under the last dot) equals 0; add 2 giving 2. The division is completed below.

$$\begin{array}{r} 0.0004553 \\ 8\dot{3}.\dot{0}\dot{4}7168\overline{)0.037812} \\ \underline{33219} \\ 4593 \\ \underline{4152} \\ 441 \\ \underline{415} \\ 26 \\ \underline{25} \\ 1 \end{array}$$

By inspection we see that if we had used 0.037810 as a dividend the only change in the problem would be after the figures 0.000455 were obtained. The remainder would then be 24 instead of 26 and the final figure in the quotient would be 2 instead of 3. This long division process is much shorter than one of the others, and it gives as much information as both of the others together.

(l) In each step from (b) to (h) we used the fact that the value of the particular polynomial equation changed sign when we substituted two values of the argument, one less, the other greater than the desired root. If the required root were a double root, we would find that the value of the polynomial would not change in sign when we substituted two values of

the argument one on each side of the required root and very nearly equal to the required root. If we separate the multiple roots as explained in section 6.9, we have only simple roots and do not have to worry about this difficulty.

(m) To find a negative root of an equation by Horner's method the student may prefer to change the signs of the roots according to the method of section 5.20 and then find the root of the resulting equation. On the other hand, if the desired root is between -4 and -3, we can add 4 to the roots of the equation and the required root of the new equation will be between 0 and 1.

(n) If two roots of the equation are very close together we can substitute three values of x: $x_1 < x_2 < x_3$; and if $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$ where $f(x_1)$, $f(x_2)$ and $f(x_3)$ are all positive, the roots are probably between x_1 and x_3. The student will find Graeffe's method very handy for cases of this sort.

6.13 Newton's Method. Newton's method of approximating the real roots of an equation is applicable to transcendental equations as well as to polynomial equations, whereas Horner's method can be used only for polynomial equations. The following discussion explains how the method works.

Let A be a root of the equation $f(x) = 0$, and let A_1 be an approximation for the root A. Newton's method consists of finding a better approximation than the given approximation. Expand $f(x)$ using Taylor's theorem for x near A_1. We have

$$f(x) = f(A_1) + (x - A_1)f'(A_1) + \frac{(x - A_1)^2}{2}f''(A_1) + \frac{(x - A_1)^3}{3!}f'''(A_1) + \cdots. \tag{6.15}$$

If we let $x = A$, this becomes

$$0 = f(A_1) + (A - A_1)f'(A_1) + \frac{(A - A_1)^2}{2}f''(A_1) + \frac{(A - A_1)^3}{3!}f'''(A_1) + \cdots. \tag{6.16}$$

Divide this through by $f'(A_1)$ and rearrange it to give

$$A = A_1 - \frac{1}{f'(A_1)}\left[f(A_1) + \frac{(A - A_1)^2}{2}f''(A_1) + \frac{(A - A_1)^3}{3!}f'''(A_1) + \cdots\right]. \tag{6.17}$$

Define A_2 by the following equation:

$$A_2 = A_1 - \frac{f(A_1)}{f'(A_1)}. \tag{6.18}$$

Now if A_1 is near A, it may be that the terms containing $(A - A_1)^2$ and higher powers are negligible. In this case A_2 and A are practically

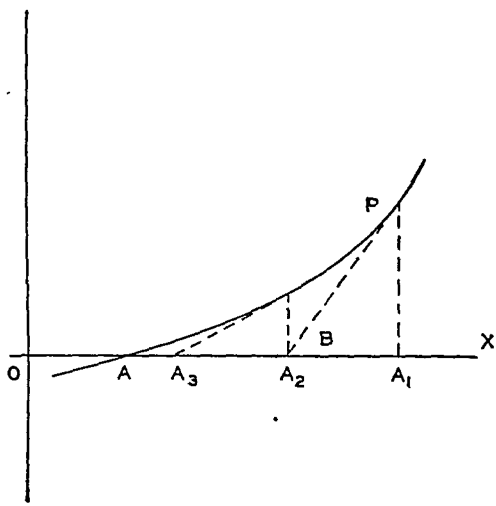

FIG. 6-1

equal and A_2 is a better approximation than A_1 Now we can continue with

$$A_3 = A_2 - \frac{f(A_2)}{f'(A_2)} \tag{6.19}$$

and A_3 is a better approximation than A_2, etc. This process can be carried on until the desired accuracy is attained.

Newton's method can be presented graphically as follows: Figure 6-1 shows $f(x)$ plotted against x for values of x near A. A_1 is the approximation to the root A. The line A_2P is tangent to the curve at $x = A_1$ and makes an angle B with the x-axis. Then

$$\tan B = f'(A_1) = \frac{f(A_1)}{A_1 - A_2}. \tag{6.20}$$

If this is solved for A_2, we have

$$A_2 = A_1 - \frac{f(A_1)}{f'(A_1)}. \tag{6.21}$$

The case illustrated in Fig. 6–1 shows that A_2 is a better approximation than A_1, and A_3 is a better approximation than A_2, etc.

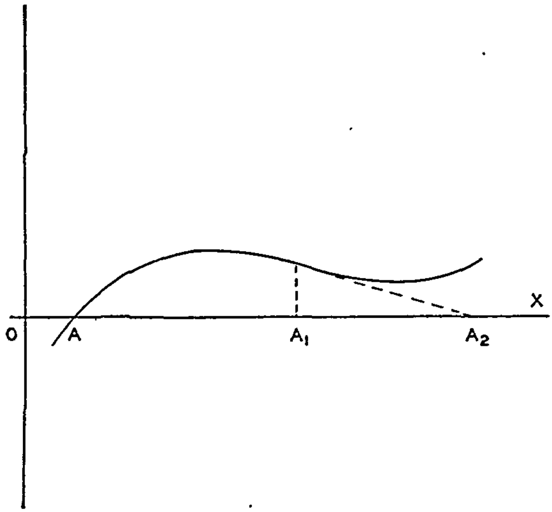

Fig. 6–2

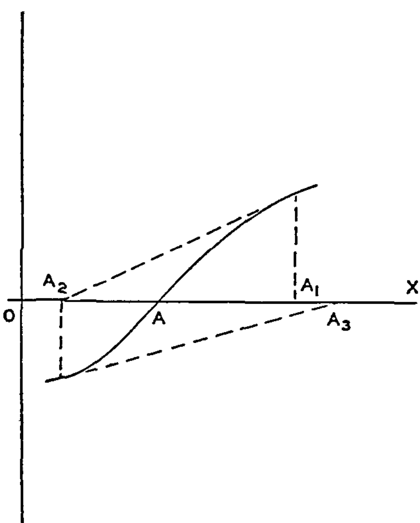

Fig. 6–3

Figure 6–2 shows a case where Newton's method fails. In this case there is a maximum to $f(x)$ between the root A and the approximation A_1.

The figure shows that A_2 is farther than A_1 from the root A. A minimum would cause the same difficulty.

A point of inflection may or may not cause trouble. Figure 6-3 shows a case where the presence of a point of inflection makes the method fail.

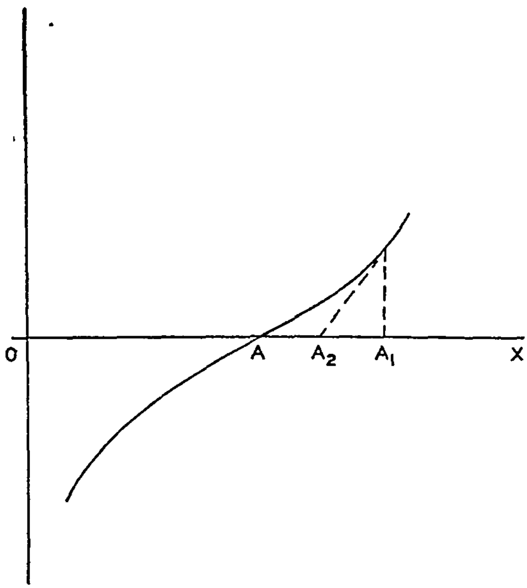

FIG. 6-4

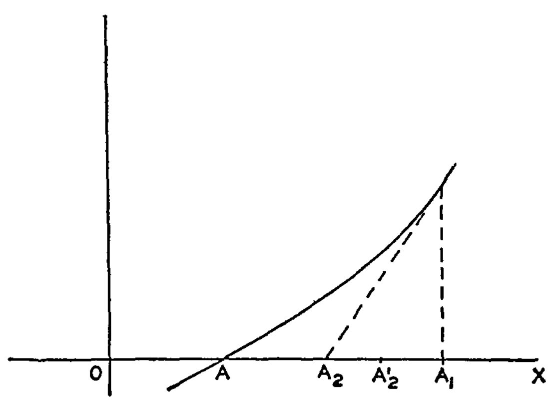

FIG. 6-5

The successive numbers A_1, A_2, A_3, etc. are alternately on one side of A, then on the other side, and there is no tendency for them to approach A. Figure 6-4 shows a case where the presence of a point of inflection does not interfere with Newton's method.

There is an interesting point in connection with Newton's method. Any

mistakes made in the first part of the work will be corrected as the work proceeds provided such mistakes are not too great. Suppose a mistake is made in computing A_2 so that A_2' is obtained instead. See Fig. 6–5. It is apparent in the figure that A_2' is a good approximation and in fact would have been the correct value if we had started with a slightly different first approximation. Of course care must be exercised in the later steps.

Example 1. Let $f(x) = x^2 - 2 = 0$ be the equation to be solved. The desired root is between 1 and 2. Let the first approximation A_1 be 1. Since $f'(x) = 2x$, we have

$$
\begin{aligned}
A_2 &= A_1 - \frac{f(A_1)}{f'(A_1)} \\
&= 1 - \frac{-1}{2} = 1.5 \\
A_3 &= A_2 - \frac{f(A_2)}{f'(A_2)} \\
&= 1.5 - \frac{2.25 - 2}{3} = 1.5 - 0.083 = 1.42 \\
A_4 &= A_3 - \frac{f(A_3)}{f'(A_3)} \\
&= 1.42 - \frac{2.0164 - 2}{2.84} = 1.42 - 0.00578 = 1.41422
\end{aligned}
$$

This answer is correct to four places.

$$
\begin{aligned}
1.41422^2 &= 2.0000182084 \\
1.41421^2 &= 1.9999899241
\end{aligned}
$$

Example 2. Let $f(x) = 2\cos x - 3x = 0$ be the given equation. $f'(x) = -2\sin x - 3$. Let the first approximation be $A_1 = 0.7$.

$$
\begin{aligned}
A_2 &= A_1 - \frac{f(A_1)}{f'(A_1)} \\
&= 0.7 - \frac{2\cos 0.7 - 2.1}{-2\sin 0.7 - 3} \\
&= 0.7 - \frac{1.530 - 2.1}{-1.288 - 3} = 0.7 - \frac{-0.570}{-4.288} \\
&= 0.7 - 0.133 = 0.567 \\
A_3 &= A_2 - \frac{f(A_2)}{f'(A_2)} \\
&= 0.567 - \frac{1.68704 - 1.701}{-1.07420 - 3} = 0.567 - \frac{0.01396}{4.0742} \\
&= 0.567 - 0.00342 = 0.56358
\end{aligned}
$$

6.14 Relations between Roots and Coefficients. When a polynomial equation is written with the coefficient of the highest power of x as unity,

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \tag{6.22}$$

a_1 is minus the sum of the roots.
a_2 is plus the sum of products of the roots taken two at a time in all possible ways.
a_3 is minus the sum of products of the roots taken three at a time in all possible ways.
a_4 s plus the sum of products of the roots taken four at a time in all possible ways, etc.
a_n is plus or minus the product of the roots, plus if n is even, minus if n is odd.

This is illustrated in the following examples: Write the equation in factored form and then multiply the factors.

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1r_2 = 0.$$

$$(x - r_1)(x - r_2)(x - r_3) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_2r_3 + r_1r_3)x - r_1r_2r_3 = 0.$$

These relations alone unfortunately will not help us solve an equation. This fact is illustrated in the following simple case. The quadratic equation $x^2 + 5x + 4 = 0$ is to be solved. Let r and s be the roots. Then $r + s = -5$ and $rs = 4$. We have two equations in two unknowns, and we can solve for r as follows: From the first equation we have $s = -r - 5$; substitute this in the second equation, $r(-r - 5) = 4$, which gives $r^2 + 5r + 4 = 0$, and we are exactly where we started. On the other hand, if we know that one root is very much larger than the other, we can make a very close approximation; this deduction leads us to Graeffe's method.

6.15 Graeffe's Method of Obtaining the Roots of an Algebraic Equation. If one root of an equation of the nth degree is much larger than all the other roots, it will be very nearly equal to minus the coefficient of x^{n-1}. Consider the equation $(x - 100)(x - 1) = x^2 - 101x + 100 = 0$. If we designate the roots by r_1 and r_2, we have

$$r_1 + r_2 = 101, \quad r_1r_2 = 100.$$

Knowing that r_1 is much larger than r_2, we say

$$r_1 = 101 \text{ approximately},$$

$$r_2 = \frac{100}{r_1} = \frac{100}{101}, \text{ approximately}$$

and we are within 1 per cent of the correct values 100 and 1.

In general, in the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

if the roots are $r_1, r_2, r_3, \cdots r_n$, and r_1 is much larger than r_2 and r_2 is much larger than $r_3, \cdots$ and r_{n-1} is much larger than r_n, we can assume that the equation is

$$x^n - r_1x^{n-1} + r_1r_2x^{n-2} - r_1r_2r_3x^{n-3} + \cdots = 0.$$

Example.

$$(x - 10{,}000)(x - 100)(x - 1) = x^3 - 10101x^2 + 1010100x - 1{,}000{,}000 = 0.$$

$$r_1 = 10{,}101 \text{ instead of } 10{,}000$$

$$r_2 = \frac{1{,}010{,}100}{10{,}101} = 100$$

$$r_3 = \frac{1{,}000{,}000}{1{,}010{,}100} = .9900 \text{ instead of } 1$$

This is all very well for the few equations whose roots are separated by large amounts, but what about the usual equation where the roots may be close together? Consider the equation

$$x^2 - 3x + 2 = 0.$$

Let us find an equation whose roots are the squares of the roots of the given equation. The simplest way to obtain it is to rewrite the equation

$$x^2 + 2 = 3x$$

and square both sides

$$x^4 + 4x^2 + 4 = 9x^2$$
$$x^4 - 5x^2 + 4 = 0$$

and substitute $y = x^2$ (later we shall find it more convenient to substitute $y = -x^2$). This operation gives

$$y^2 - 5y + 4 = 0.$$

The roots of this equation are 4 and 1, the squares of the roots of the preceding equation, 2 and 1. If this process is repeated an equation will be obtained with roots 16 and 1, then 256 and 1, and it is evident that one root is becoming much larger than the other, and shortly the approximation mentioned above can be used.

In the following table we have tabulated the equations obtained by performing this process and then indicated the roots obtained by the above approximation at each step.

TABLE VI-4

$x^2 - 3x + 2 = 0$	$r_1 = 3$	$r_2 = 0.667$
$y^2 - 5y + 4 = 0$	$r_1 = \sqrt{5} = 2.236$	$r_2 = \sqrt{.8} = 0.894$
$z^2 - 17z + 16 = 0$	$r_1 = \sqrt[4]{17} = 2.031$	$r_2 = \sqrt[4]{\frac{16}{17}} = 0.985$
$w^2 - 257w + 256 = 0$	$r_1 = \sqrt[8]{257} = 2.001$	$r_2 = \sqrt[8]{\frac{256}{257}} = 0.9995$

It is evident that good approximations can be made quite soon with this process. The aim now is to develop a rule so that the steps can be carried out with as little chance of error as possible.

Consider the equation in standard form

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \cdots + a_{n-1}x + a_n = 0. \qquad (6.23)$$

Transpose all the terms containing odd powers of x to one side and we have

$$-[x^n + a_2x^{n-2} + a_4x^{n-4} + \cdots] = a_1x^{n-1} + a_3x^{n-3} + \cdots. \qquad (6.24)$$

Now square both sides of equation (6.24).

$$\begin{aligned} x^{2n} + a_2^2x^{2n-4} + 2a_2x^{2n-2} + 2a_4x^{2n-4} + a_4^2x^{2n-8} + \cdots &= a_1^2x^{2n-2} \\ + a_3^2x^{2n-6} + 2a_1a_5x^{2n-6} + 2a_1a_3x^{2n-4} + \cdots&. \end{aligned} \qquad (6.25)$$

Rearranging this, we have

$$\begin{aligned} x^{2n} - x^{2n-2}(a_1^2 - 2a_2) + x^{2n-4}(a_2^2 - 2a_1a_3 + 2a_4)& \\ - x^{2n-6}(a_3^2 - 2a_2a_4 + 2a_1a_5 - 2a_6) + \cdots = 0.& \end{aligned} \qquad (6.26)$$

We now substitute $-y$ for x^2 and equation (6.26) becomes

$$y^n + y^{n-1}(a_1^2 - 2a_2) + y^{n-2}(a_2^2 - 2a_1a_3 + 2a_4) + \cdots = 0. \qquad (6.27)$$

If we now tabulate the coefficients of the x-equation and place the coefficients of the y-equation under the corresponding coefficients of the x-equation, the general rule will become evident.

TABLE VI-5

x	1	a_1	a_2	a_3	a_4	$a_5 \cdots$
y	1	a_1^2 $-2a_2$	a_2^2 $-2a_1a_3$ $+2a_4$	a_3^2 $-2a_2a_4$ $+2a_1a_5$ $-2a_6$	a_4^2 $-2a_3a_5$ $+2a_2a_6$ $-2a_1a_7$ $+2a_8$	a_5^2 $-2a_4a_6$ $+2a_3a_7$ $-2a_2a_8$ $+2a_1a_9$ $-2a_{10}$

The coefficient of a given term in the y-equation is the square of the coefficient of the same term in the x-equation minus twice the product of the coefficients of the terms on each side in the x-equation plus twice the product of the coefficients of the terms on each side of the three already considered, etc. If the x-equation is of the fourth degree then a_5, a_6, a_7, a_8, a_9, and a_{10} in the table will all be zero.

It is desirable at this point to introduce several examples to show how far to carry the root squaring process and also to explain how equal roots and non-real roots make themselves known during the process.

Example 1. Required the roots of $x^3 - 35x^2 - 206x + 240 = 0$. The work is carried out in Table VI–6. It is evident that the coefficients in each column approach the squares of the preceding coefficients, e.g., 2.56×10^6 is nearly the square of 1.637×10^3. When this relation is satisfied within satisfactory limits the process is terminated and the roots are obtained as indicated under the table. There is still a question as to whether the roots should be positive or negative. This question can be decided by substituting in the original equation. We find in this case that the roots are 1.0009, −5.997, and 39.99, or 1, −6, and 40.

TABLE VI–6

	x^3	x^2	x^1	x^0
1	1	−35	−206	240
	1	1.225×10^3 0.412×10^3	4.24×10^4 1.68×10^4	5.76×10^4
2	1	1.637×10^3	5.92×10^4	5.76×10^4
	1	2.68×10^6 -0.12×10^6	3.50×10^9 -0.19×10^9	3.32×10^9
4	1	2.56×10^6	3.31×10^9	3.32×10^9
	1	6.55×10^{12} -0.006×10^{12}	1.096×10^{19} -0.0008×10^{19}	1.102×10^{19}
8	1	6.54×10^{12}	1.095×10^{19}	1.102×10^{19}

$$|r_1| = \sqrt[8]{6.54 \times 10^{12}} = 39.99$$

$$|r_2| = \sqrt[8]{\frac{1.095 \times 10^{19}}{6.54 \times 10^{12}}} = 5.997$$

$$|r_3| = \sqrt[8]{\frac{1.102 \times 10^{19}}{1.095 \times 10^{19}}} = 1.0009$$

If we skip a coefficient in the last line of the table we get the product of two roots.

$$|r_1r_2| = \sqrt[8]{1.095 \times 10^{19}}$$

$$|r_2r_3| = \sqrt[8]{\frac{1.102 \times 10^{19}}{6.54 \times 10^{12}}}$$

If we skip two consecutive coefficients we get the product of three roots, etc.

6.16 Non-real Roots. Suppose r_2 and r_3 are a conjugate pair of non-real roots. Then the coefficient of x^{n-2} will approach after k steps in the root squaring process $r_1^m r_2^m + r_1^m r_3^m$ instead of $r_1^m r_2^m$ where $m = 2^k$. The coefficient of x^{n-3} will approach $-r_1^m r_2^m r_3^m$ and the other coefficients will follow the usual pattern. Let

$$r_2 = z\, e^{i\phi}, \qquad r_3 = z\, e^{-i\phi}. \tag{6.28}$$

The coefficient of x^{n-2} will approach

$$r_1^m(r_2^m + r_3^m) = r_1^m z^m(e^{im\phi} + e^{-im\phi}) \tag{6.29}$$

$$= 2r_1^m z^m \cos m\phi. \tag{6.30}$$

As k is increased, $\cos m\phi = \cos 2^k\phi$ will at times be plus and at times be minus (except, possibly for the special case considered in the next section). Therefore an irregular alternation in sign of one coefficient indicates non-real roots. If we skip this coefficient we get the product of the pair of roots which will be a real number because the roots are conjugate to one another.

$$(z\, e^{i\phi})(z\, e^{-i\phi}) = z^2. \tag{6.31}$$

We are able to obtain the modulus of each complex root of the equation by following these straightforward rules. To determine the argument ϕ, and thereby completely determine the roots, requires the use of one or more of the equations in section 6.14 which states the relations between the roots and the coefficients. In the first example that follows there is one pair of conjugate roots and we need only one of these relations.

Let the non-real roots be $u + iv$ and $u - iv$. If all the other roots are real they can be determined as directed above. The coefficient of x^{n-1} in the original equation is minus the sum of the roots or, in this case,

$$-(2u + \text{sum of real roots}).$$

This relation gives us u. We then get v from

$$v^2 = z^2 - u^2. \tag{6.32}$$

The coefficient of a given term in the y-equation is the square of the coefficient of the same term in the x-equation minus twice the product of the coefficients of the terms on each side in the x-equation plus twice the product of the coefficients of the terms on each side of the three already considered, etc. If the x-equation is of the fourth degree then a_5, a_6, a_7, a_8, a_9, and a_{10} in the table will all be zero.

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	1	2.68×10^6 -0.12×10^6	3.50×10^9 -0.19×10^9	3.32×10^9
4	1	$2\ 56 \times 10^6$	3.31×10^9	3.32×10^9
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$$|r_3| = \sqrt[8]{\frac{1.102 \times 10^{19}}{1.095 \times 10^{19}}} = 1.0009$$

If we skip a coefficient in the last line of the table we get the product of two roots.

$$|r_1r_2| = \sqrt[8]{1.095 \times 10^{19}}$$

$$|r_2r_3| = \sqrt[8]{\frac{1.102 \times 10^{19}}{6.54 \times 10^{12}}}$$

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$$r_2 = z\, e^{i\phi}, \qquad r_3 = z\, e^{-i\phi}. \tag{6.28}$$

The coefficient of x^{n-2} will approach

$$r_1^m(r_2^m + r_3^m) = r_1^m z^m(e^{im\phi} + e^{-im\phi}) \tag{6.29}$$

$$= 2r_1^m z^m \cos m\phi. \tag{6.30}$$

As k is increased, $\cos m\phi = \cos 2^k\phi$ will at times be plus and at times be minus (except, possibly for the special case considered in the next section). Therefore an irregular alternation in sign of one coefficient indicates non-real roots. If we skip this coefficient we get the product of the pair of roots which will be a real number because the roots are conjugate to one another.

$$(z\, e^{i\phi})(z\, e^{-i\phi}) = z^2. \tag{6.31}$$

We are able to obtain the modulus of each complex root of the equation by following these straightforward rules. To determine the argument ϕ, and thereby completely determine the roots, requires the use of one or more of the equations in section 6.14 which states the relations between the roots and the coefficients. In the first example that follows there is one pair of conjugate roots and we need only one of these relations.

Let the non-real roots be $u + iv$ and $u - iv$. If all the other roots are real they can be determined as directed above. The coefficient of x^{n-1} in the original equation is minus the sum of the roots or, in this case,

$$-(2u + \text{sum of real roots}).$$

This relation gives us u. We then get v from

$$v^2 = z^2 - u^2. \tag{6.32}$$

Example. Required the roots of $x^3 + 94x^2 - 575x + 2500 = 0$.

TABLE VI-7

	x^3	x^2	x	x^0
1	1	94	-575	2500
	1	8.836×10^3 1.150×10^3	3.306×10^5 -4.700×10^5	6.25×10^6
2	1	9.986×10^3	-1.394×10^5	6.25×10^6
	1	9.972×10^7 0.028×10^7	1.943×10^{10} -12.483×10^{10}	3.906×10^{13}
4	1	10.000×10^7	-10.540×10^{10}	3.906×10^{13}

$$|r_1| = \sqrt[4]{10^8} = 100$$

$$z = \sqrt[8]{\frac{3.906 \times 10^{13}}{10^8}} = 4.999$$

Substitution shows that $r_1 = -100$

Now:

$$-94 = -100 + 2u$$

$$u = 3$$

$$v = \sqrt{4.999^2 - 3^2} = 4$$

The roots are -100, $3 + i4$, $3 - i4$.

Note that we skip the coefficient in the x column above because it does not follow the usual pattern of approaching the square of the preceding coefficient. The coefficient in any column which does not behave according to the usual pattern is always skipped and we compute the product of two or more roots depending on whether one isolated column or two or more adjacent columns are misbehaving.

Example. Determine the roots of

$$x^5 + 3x^4 - 4x^3 - 28x^2 + 43x + 65 = 0$$

TABLE VI-8

	x^5	x^4	x^3	x^2	x	x^0
1	1	3	−4	−28	43	65
	1	9 8	16 168 86	7.86×10^2 3.44×10^2 3.90×10^2	1.85×10^3 3.64×10^3	4.22×10^3
2	1	1.70×10	2.70×10^2	1.510×10^3	5.49×10^3	4.22×10^3
	1	2.89×10^2 -5.40×10^2	7.29×10^4 -5.13×10^4 1.10×10^4	2.26×10^6 -2.96×10^6 0.14×10^6	3.01×10^7 -1.27×10^7	1.78×10^7
4	1	-2.51×10^2	3.26×10^4	-5.60×10^5	1.74×10^7	1.78×10^7
	1	6.30×10^4 -6.52×10^4	1.063×10^9 -0.28×10^9 0.03×10^9	3.14×10^{11} -11.34×10^{11} -0.09×10^{11}	3.03×10^{14} 0.10×10^{14}	3.17×10^{14}
8	1	-0.22×10^4	8.1×10^8	-8.29×10^{11}	3.13×10^{14}	3.17×10^{14}

$$|x_5| = \sqrt[8]{\frac{3.17 \times 10^{14}}{3.13 \times 10^{14}}} = 1$$

By substitution we find $x_5 = -1$

Let

$$x_1 = u_1 + iv_1 = z_1 e^{i\phi_1} \tag{6.33}$$

$$x_2 = u_1 - iv_1 = z_1 e^{-i\phi_1} \tag{6.34}$$

$$x_3 = u_2 + iv_2 = z_2 e^{i\phi_2} \tag{6.35}$$

$$x_4 = u_2 - iv_2 = z_2 e^{-i\phi_2} \tag{6.36}$$

The coefficient of x^4 in the original equation gives

$$-3 = x_1 + x_2 + x_3 + x_4 + x_5 \tag{6.37}$$

$$-3 = 2u_1 + 2u_2 - 1 \tag{6.38}$$

Therefore

$$u_1 + u_2 = -1 \tag{6.39}$$

From Table VI-8 we have

$$x_1x_2 = z_1^2 = \sqrt[8]{8.1 \times 10^8} = 13 \tag{6.40}$$

$$x_3x_4 = z_2^2 = \sqrt[8]{\frac{3.13 \times 10^{14}}{8.1 \times 10^8}} = 5 \tag{6.41}$$

The coefficient of x in the original equation gives

$$43 = x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 \tag{6.42}$$

$$= z_1^2z_2^2 - z_1^2x_3 - z_1^2x_4 - z_2^2x_1 - z_2^2x_2 \tag{6.43}$$

$$= z_1^2z_2^2 - z_1^2(x_3 + x_4) - z_2^2(x_1 + x_2) \tag{6.44}$$

$$= z_1^2z_2^2 - 2u_2z_1^2 - 2u_1z_2^2 \tag{6.45}$$

$$= (13)(5) - 26u_2 - 10u_1 \tag{6.46}$$

$$26u_2 + 10u_1 = 22 \tag{6.47}$$

This with equation (6.39) enables us to solve for u_1 and u_2.

$$u_2 = 2 \qquad u_1 = -3$$

$$v_2 = \sqrt{z_2^2 - u_2^2} = \sqrt{5 - 4} = 1$$

$$v_1 = \sqrt{z_1^2 - u_1^2} = \sqrt{13 - 9} = 2$$

The roots of the equation are

$$\begin{aligned} x_1 &= -3 + i2 \\ x_2 &= -3 - i2 \\ x_3 &= 2 + i \\ x_4 &= 2 - i \\ x_5 &= -1 \end{aligned}$$

6.17 Double Roots and Roots of the Type r cis $2^{-k}\pi$. If two roots of an equation are equal (if the equation has a double root), no amount of root squaring will separate them. Suppose the second and third largest roots are equal, $r_1 > r_2 = r_3 > r_4 \cdots$. Then after the roots have been squared k times, the coefficient of x^{n-1} will be nearly $-r_1^m$ where $m = 2^k$, the coefficient of x^{n-2} will be nearly $2r_1^mr_2^m$ instead of $r_1^mr_2^m$, the coefficient of x^{n-3} will be nearly $r_1^mr_2^{2m}$, and that of x^{n-4} will be $r_1^mr_2^{2m}r_4^m$, etc.

The difference between this and the first case is that one column misbehaves since successive coefficients approach half the square of the preceding coefficient. We skip this column and get the product of the two roots which are equal, i.e., the square of the double root. The presence of a triple root would cause two adjacent columns to misbehave, etc.

If the multiple roots of the equation have been isolated (see section 6.9), symmetric roots would make themselves known by the same sign because the first step in the root squaring process causes a pair of symmetric roots to become a double root. The symmetric roots can be isolated using the method of section 6.10. If the equation has a pair of complex roots $1 \pm i$, the first step in Graeffe's method will make them $\pm i2$. The second step in the root squaring process will give a double root equal to -4. In fact any pair of complex roots of the form r cis$(\pm 2^{-k}\pi)$ will turn into a double root after enough steps in the root squaring process. If multiple roots

have been removed, and symmetric roots have been removed, and a double root turns up, it should be tested as such a complex pair instead of assuming that the work has been incorrect.

The following example shows how Graeffe's method appears when there is a double root. Required the roots of $x^3 - 204x^2 + 804x - 800 = 0$.

TABLE VI-9

	x^3	x^2	x	x^0
1	1	-204	804	-800
	1	4.162×10^4 -0.161×10^4	6.464×10^5 -3.264×10^5	6.4×10^5
2	1	4.001×10^4	3.200×10^5	6.4×10^5
	1	1.601×10^9 -0.0006×10^9	1.024×10^{11} -0.512×10^{11}	4.096×10^{11}
4	1	1.600×10^9	0.512×10^{11}	4.096×10^{11}

1.6×10^9 is practically the square of 4.001×10^4, 4.096×10^{11} is of course the square of 6.4×10^5, but 0.512×10^{11} is only half the square of 3.2×10^5. This indicates that the smaller roots are equal. The roots are now found numerically from the next to the last step.

$$|r_1| = \sqrt{4.001 \times 10^4} = 200.025$$

$$|r_2| = |r_3| = \sqrt[4]{\frac{6.4 \times 10^{15}}{4.001 \times 10^4}} = 1.99988.$$

Substitution in the original equation shows that the roots are all positive. Note that we do not use 3.2×10^5 which appears in the column in which successive coefficients do not become the squares of the preceding coefficients.

6.18 Special Case of the Quadratic. The quadratic equation

$$ax^2 + bx + c = 0 \tag{6.48}$$

can be solved to any required degree of accuracy by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{6.49}$$

If $4ac$ is small compared to b^2, one root is small and the computations must be carried to many decimal places to get anything but zero for that root.

CHAPTER 7

FOURIER SERIES

7.1 Introduction. The voltage of an alternator, the output torque of a diesel engine, and the load torque of a reciprocating pump are all periodic functions of time. The manner in which the output torque of an engine varies in time can be approximated during a certain time interval by a polynomial; if the expression is to apply to a large time interval we may have to replace the polynomial with an infinite series.

If a quantity to be approximated is a periodic function of x, it is a reasonable suggestion to use a polynomial in periodic functions. This is possible but it is more convenient to use $\sin x$, $\sin 2x$, $\sin 3x$, $\cdots$ instead of $\sin x$, $\sin^2 x$, $\sin^3 x$, $\cdots$ and to use $\cos x$, $\cos 2x$, $\cos 3x$, $\cdots$ too. Furthermore, we expect the approximation to be better as more terms are used, just as higher degree terms in a polynomial generally enable us to make a better approximation to a given function in a fixed interval.

Assume that we have given the curve $y = f(x)$. By assigning the correct values to A_0, a_1, and b_1 we can make the curve $y = A_0 + a_1 \cos x + b_1 \sin x$ intersect the given curve in three points. To find what values are required to make the two curves intersect at x_1, x_2, and x_3 we need only solve the following three equations for A_0, a_1, and b_1.

$$\begin{aligned} A_0 + a_1 \cos x_1 + b_1 \sin x_1 &= f(x_1) \\ A_0 + a_1 \cos x_2 + b_1 \sin x_2 &= f(x_2) \\ A_0 + a_1 \cos x_3 + b_1 \sin x_3 &= f(x_3) \end{aligned}$$

The curve of $y = A_0 + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$ can be made to intersect the given curve $y = f(x)$ in five points. The curve of

$$\begin{aligned} y = A_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx + b_1 \sin x + b_2 \sin 2x \\ + \cdots + b_n \sin nx \end{aligned}$$

can be made to intersect the given curve $y = f(x)$ in $2n + 1$ points.

If we let n become infinite then we might ask if the curve of $y = A_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots$ can be made to intersect the given curve $y = f(x)$ in an infinite number of points, say every point from $x = 0$ to $x = 2\pi$.

7.2 Fourier's Theorem. Any single valued function, $y = f(x)$, defined throughout the interval from $x = 0$ to $x = 2\pi$, which does not have an

infinite number of discontinuities and which does not have an infinite number of maxima or minima, can be represented by a Fourier series which is a series of the form

$$y = A_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots \tag{7.1}$$

where the coefficients are defined by the formulas

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx, \tag{7.2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx, \tag{7.3}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx. \tag{7.4}$$

A proof of Fourier's theorem can be found in an advanced mathematics text.

7.3 Derivation of Formulas. Let us assume that the function $y = f(x)$ is one which can be represented by a Fourier series. Then we can write

$$f(x) = A_0 + a_1 \cos x + a_2 \cos 2x + \cdots + b_1 \sin x + b_2 \sin 2x + \cdots.$$

Assume that we can integrate the series term by term; then, if we integrate from $x = 0$ to $x = 2\pi$, every term on the right will be zero except the first, and we have

$$\int_0^{2\pi} f(x)\, dx = \int_0^{2\pi} A_0\, dx = 2\pi A_0 \tag{7.5}$$

or

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx. \tag{7.6}$$

Now suppose we multiply both sides by $\cos nx$ before we integrate. On the right we have terms of the following types:

$$\int_0^{2\pi} A_0 \cos nx\, dx = 0, \tag{7.7}$$

$$\int_0^{2\pi} a_m \cos mx \cos nx\, dx$$

$$= \frac{a_m}{2}\int_0^{2\pi} [\cos (m - n)x + \cos (m + n)x]\, dx = 0 \qquad (m \neq n), \tag{7.8}$$

$$\int_0^{2\pi} a_n \cos^2 nx\, dx = \frac{a_n}{2}\int_0^{2\pi} (1 + \cos 2nx)\, dx = \pi a_n, \tag{7.9}$$

$$\int_0^{2\pi} b_n \sin mx \cos nx \, dx$$
$$= \frac{b_m}{2} \int_0^{2\pi} [\sin (m + n)x + \sin (m - n)x] \, dx = 0 \qquad (m \neq n), \quad (7.10)$$

$$\int_0^{2\pi} b_n \sin nx \cos nx \, dx = \frac{b_n}{2} \int_0^{2\pi} \sin 2nx \, dx = 0. \quad (7.11)$$

Every term on the right will be zero but one, giving

$$\int_0^{2\pi} f(x) \cos nx \, dx = \pi a_n, \quad (7.12)$$

thus proving equation (7.3). If we substitute $n = 0$ in equation (7.3), we have

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \, dx, \quad (7.13)$$

which is just twice A_0. We shall always use $0.5a_0$ instead of A_0 from here on. Therefore we no longer require equation (7.2). The proof of equation (7.4) is left as an exercise for the student.

If the function $f(x)$ is periodic, such that $f(x) \equiv f(x + 2\pi)$, any pair of limits can be used on the integrals in equations (7.2), (7.3), and (7.4) so long as the difference of the limits is equal to 2π. It is sometimes more convenient to integrate from $-\pi$ to $+\pi$ instead of from 0 to 2π. This is especially true when the function $f(x)$ has some symmetry. This matter is considered later in the chapter.

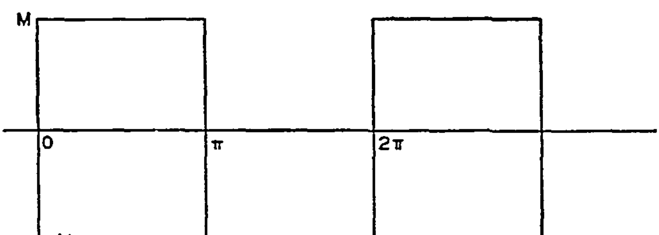

Fig. 7–1

7.4 Examples of Fourier Series. Suppose the function to be represented by the Fourier series has the form of Fig. 7–1. This is a rectangular wave and can be defined by the following equations:

$$\begin{aligned} y &= f(x) = M, & 0 < x < \pi, \\ y &= f(x) = -M, & \pi < x < 2\pi. \end{aligned} \quad (7.14)$$

Substitute the value of $f(x)$ given in equations (7.14) in equation (7.3). We leave n unspecified as long as possible so that we may find a_n in terms of n.

$$a_n = \frac{1}{\pi}\int_0^{\pi} y \cos nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} y \cos nx\, dx \tag{7.15}$$

$$= \frac{1}{\pi}\int_0^{\pi} M \cos nx\, dx - \frac{1}{\pi}\int_{\pi}^{2\pi} M \cos nx\, dx \tag{7.16}$$

$$= -\frac{M}{\pi n}\sin nx\,\Big]_0^{\pi} + \frac{M}{\pi n}\sin nx\,\Big]_{\pi}^{2\pi} = 0 \tag{7.17}$$

unless $n = 0$. If $n = 0$, equation (7.16) will call for the integral of a constant instead of the integral of a cosine function. We substitute $n = 0$ in equation (7.16) and have

$$a_0 = \frac{1}{\pi}\int_0^{\pi} M\, dx - \frac{1}{\pi}\int_{\pi}^{2\pi} M\, dx \tag{7.18}$$

$$= \frac{M}{\pi}\, x\Big]_0^{\pi} - \frac{M}{\pi}\, x\Big]_{\pi}^{2\pi} = 0. \tag{7.19}$$

Therefore the Fourier series representing the function in Fig. 7-1 contains no constant term and contains no cosine terms. We have still to determine b_n for the sine terms.

$$b_n = \frac{1}{\pi}\int_0^{\pi} M \sin nx\, dx - \frac{1}{\pi}\int_{\pi}^{2\pi} M \sin nx\, dx \tag{7.20}$$

$$= \frac{M}{\pi n}\Big[-\cos nx\Big]_0^{\pi} - \frac{M}{\pi n}\Big[-\cos nx\Big]_{\pi}^{2\pi} \tag{7.21}$$

$$= \frac{2M}{\pi n}(-\cos n\pi + 1). \tag{7.22}$$

The value of b_n can be obtained from equation (7.22) more easily if we first distinguish between even and odd values of n. We have, then, from equation (7.22)

$$b_n = 0 \qquad \text{for } n \text{ even},$$

$$b_n = \frac{4M}{\pi n} \qquad \text{for } n \text{ odd}. \tag{7.23}$$

The Fourier series representing the function in Fig. 7-1 is

$$y = \frac{4M}{\pi}\sin x + \frac{4M}{\pi}\frac{\sin 3x}{3} + \frac{4M}{\pi}\frac{\sin 5x}{5} + \cdots. \tag{7.24}$$

If the student will draw the curves of several of the sine functions in equation (7.24) he will see how the sum of the sine functions approaches the function given in Fig. 7–1.

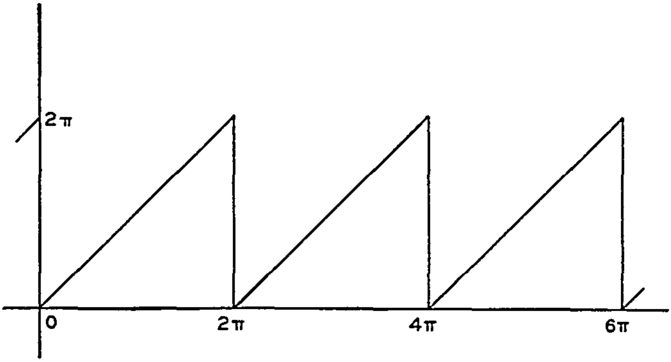

FIG 7–2

As a second example of a Fourier series consider the function illustrated in Fig. 7–2. In this case the function can be defined in the interval from 0 to 2π with just one equation

$$y = x, \qquad 0 < x < 2\pi. \tag{7.25}$$

As before, we solve for the coefficients using equations (7.3) and (7.4). We have for the cosine coefficients

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \cos nx \, dx \tag{7.26}$$

$$= \frac{1}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{2\pi} = 0, \qquad n \neq 0. \tag{7.27}$$

Since n appears in the denominator in equation (7.27), we must evaluate a_0 as a special case

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \, dx = \frac{1}{\pi}\frac{x^2}{2}\bigg]_0^{2\pi} = 2\pi. \tag{7.28}$$

The Fourier series contains no cosine terms but does contain a constant term equal to π. To find the sine terms,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin nx \, dx \tag{7.29}$$

$$= \frac{1}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi} = -\frac{2}{n}. \tag{7.30}$$

The Fourier series for the function of Fig. 7–2 is

$$y = \pi - 2\sin x - \sin 2x - \tfrac{2}{3}\sin 3x - \tfrac{1}{2}\sin 4x - \cdots. \qquad (7.31)$$

7.5 Discontinuities. If we substitute $x = 0$ or $x = \pi$ in equation (7.24), we find $y = 0$. The value 0 is halfway between M and $-M$. If we substitute $x = 0$ or $x = 2\pi$ in equation (7.31), we find $y = \pi$ which is halfway between 0 and 2π. If the function is discontinuous at a point x_1, and if x_1 is substituted in the Fourier series, the resulting value of y determined by the Fourier series will be the average of the two values approached as x approaches x_1 from above and from below.

7.6 Other Forms of Fourier Series. The trigonometric identity $\sin(a + b) = \sin a \cos b + \cos a \sin b$ enables us to write the Fourier series in another form which is useful at times. The Fourier series was written in equation (7.1) in terms of sines and cosines.

$$\begin{aligned} f(x) = 0.5a_0 &+ a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \\ &+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots. \end{aligned} \qquad (7.1)$$

We also have

$$\begin{aligned} f(x) = 0.5a_0 &+ c_1 \sin(x + \alpha_1) + c_2 \sin(2x + \alpha_2) \\ &+ c_3 \sin(3x + \alpha_3) + \cdots \end{aligned} \qquad (7.32)$$

where

$$c_n^2 = a_n^2 + b_n^2$$

and

$$\tan \alpha_n = \frac{a_n}{b_n}.$$

Similarly, $\cos(a - b) = \cos a \cos b + \sin a \sin b$ enables us to write the Fourier series in terms of cosines alone.

$$\begin{aligned} f(x) = 0.5a_0 &+ c_1 \cos(x - \beta_1) + c_2 \cos(2x - \beta_2) \\ &+ c_3 \cos(3x - \beta_3) + \cdots \end{aligned} \qquad (7.33)$$

where

$$c_n^2 = a_n^2 + b_n^2$$

and

$$\tan \beta_n = \frac{b_n}{a_n}.$$

In determining the values of the α's or β's it is necessary that the correct quadrant be determined. For example, tan 110° is equal to tan −70°, but one will be incorrect. If the angle should be 110° then −70° would be incorrect.

7.7 Periodicity. The Fourier series as written in equation (7.1) is periodic, and the period is 2π. If a periodic function with a period other than 2π is to be represented we need only change the independent variable to make formulas (7.3) and (7.4), developed above, applicable. For example, if the voltage of an alternator varies in time, t, with a period T, to transform from the independent variable, x, with a period 2π, we use the relation

$$\frac{t}{T} = \frac{x}{2\pi}. \tag{7.34}$$

The Fourier series now becomes

$$\begin{aligned} F(t) &= 0.5a_0 + a_1 \cos\frac{2\pi t}{T} + a_2 \cos\frac{4\pi t}{T} + a_3 \cos\frac{6\pi t}{T} + \cdots \\ &\quad + b_1 \sin\frac{2\pi t}{T} + b_2 \sin\frac{4\pi t}{T} + b_3 \sin\frac{6\pi t}{T} + \cdots. \end{aligned} \tag{7.35}$$

In place of equations (7.3) and (7.4), we can use

$$a_n = \frac{2}{T}\int_0^T F(t)\cos\frac{2n\pi t}{T}\,dt, \tag{7.36}$$

$$b_n = \frac{2}{T}\int_0^T F(t)\sin\frac{2n\pi t}{T}\,dt. \tag{7.37}$$

A common way of writing the series is in terms of the reciprocal of T. Designate the reciprocal of T by f. Then equation (7.35) becomes

$$\begin{aligned} F(t) &= 0.5a_0 + a_1 \cos 2\pi ft + a_2 \cos 4\pi ft + a_3 \cos 6\pi ft + \cdots \\ &\quad + b_1 \sin 2\pi ft + b_2 \sin 4\pi ft + b_3 \sin 6\pi ft + \cdots. \end{aligned} \tag{7.38}$$

If we now substitute $2\pi f = \omega$, we have

$$\begin{aligned} F(t) &= 0.5a_0 + a_1 \cos \omega t + a_2 \cos 2\omega t + a_3 \cos 3\omega t + \cdots \\ &\quad + b_1 \sin \omega t + b_2 \sin 2\omega t + b_3 \sin 3\omega t + \cdots. \end{aligned} \tag{7.39}$$

It is possible to represent nonperiodic functions with Fourier series if the following precautions are kept in mind. If $y = f(x)$ is a nonperiodic function which is to be used for only a finite range of values of x, this range of values of x can be assumed as the period and the function can be treated as was done above. Consider the straight line $y = x$. If the only points on the line that concern us are points between 0 and 2π, the series in equation (7.31) will represent the function in that interval. If we keep x within the interval from 0 to 2, we can still use the series in equation (7.31), but if x is allowed to range in the interval from 0 to 4, equation (7.31) will not

give the correct values for y when x is between 2π and 4. In this case we would have to make a new analysis with the period not less than 4.

As an illustration of the procedure we shall work the problem where $y = x$ in the interval $0 < x < 4$ is to be represented by a Fourier series. We shall let the period be 4. Note that for $x = 0$ and for $x = 4$ the value of y will be 2.

$$a_n = \frac{2}{4}\int_0^4 x \cos\frac{2\pi nx}{4}\,dx = 0, \qquad \text{for } n \neq 0, \tag{7.40}$$

$$a_0 = \frac{2}{4}\int_0^4 x\,dx = 4, \tag{7.41}$$

$$b_n = \frac{2}{4}\int_0^4 x \sin\frac{2\pi nx}{4}\,dx = -\frac{4}{n\pi}. \tag{7.42}$$

The function $y = x$ in the interval $0 < x < 4$ can be represented by the Fourier series

$$y = 2 - \frac{4}{\pi}\sin\frac{\pi x}{2} - \frac{2}{\pi}\sin \pi x - \frac{4}{3\pi}\sin\frac{3\pi x}{2} - \frac{1}{\pi}\sin 2\pi x - \cdots. \tag{7.43}$$

7.8 Symmetry. Frequently a function to be represented by a Fourier series has some type of symmetry that can be used to simplify the mathematical treatment. We consider only two types of symmetry in this chapter. If a function is symmetrical with respect to the y axis, or if $f(x) \equiv f(-x)$, the function is called an even function. If the function is symmetrical with respect to the origin, or if $f(x) \equiv -f(-x)$, the function is called an odd function.

MacLaurin's series for the sine and cosine are

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \tag{7.44}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots. \tag{7.45}$$

The sine function is represented by a series containing odd powers of x and is an odd function; therefore, $\sin x \equiv -\sin -x$. The cosine contains even powers of x including the zero power and is an even function and $\cos x \equiv \cos -x$. The terms odd and even in this case apply to the exponents of the variable.

A little reflection will show that an odd function can be represented only by odd functions and an even function only by even functions. Therefore if a function has the symmetry $f(x) \equiv f(-x)$ it is an even function and the Fourier series contains no sine terms and $b_n = 0$. On the other hand, if a

function has the symmetry $f(x) \equiv -f(-x)$, it is an odd function and $a_n = 0$ for all values of n, and the series contains nothing but sine terms.

If a function satisfies either of the symmetries described above we can obtain the desired coefficients by integrating from 0 to π in formulas (7.3) and (7.4) and using twice the result. This can be shown by the following argument. If $f(x)$ is an even function $f(x)\cos nx$ is even, if $f(x)$ is odd $f(x)\sin nx$ is even; in either case the integrand is even. Call it $F(x)$ and we have

$$Q = \int_{-\pi}^{\pi} F(x)\,dx = \int_{-\pi}^{0} F(x)\,dx + \int_{0}^{\pi} F(x)\,dx, \tag{7.46}$$

$$Q = -\int_{0}^{-\pi} F(x)\,dx + \int_{0}^{\pi} F(x)\,dx. \tag{7.47}$$

Replace x in the first integral above by $-x$ and adjust the upper limit,

$$Q = \int_{0}^{\pi} F(-x)\,dx + \int_{0}^{\pi} F(x)\,dx. \tag{7.48}$$

But $F(x)$ is even; therefore $F(-x) \equiv F(x)$ and we have

$$Q = \int_{0}^{\pi} F(x)\,dx + \int_{0}^{\pi} F(x)\,dx = 2\int_{0}^{\pi} F(x)\,dx. \tag{7.49}$$

Note that when we refer to odd and even functions we do not imply anything in regard to the value of n or the order of the harmonics in the Fourier series. An odd function may have even sine harmonics in the Fourier series as well as odd harmonics, and an even function may have odd cosine harmonics as well as even cosine harmonics.

7.9 Sine and Cosine Series. When a nonperiodic function is to be represented we can usually represent it by more than one type of Fourier series. If the interval of values of the independent variable in which the function is defined is taken as the period, we obtain a series that may have sine or cosine terms or both sine and cosine terms, depending upon the particular function. A function may be represented by a sine series or a cosine series if the interval of definition is made half the period and the function is arbitrarily defined in the other half period to make the periodic function odd or even.

Figure 7–3 shows the function $y = x$. This function in the interval $0 < x < 4$ is represented by the Fourier series in equation (7.43). The Fourier series in equation (7.43) really represents the periodic function of

Fig. 7–4. We may say that we have changed the function $y = x$ into the periodic function in Fig. 7–4. We can change the function in Fig. 7–3 into any other periodic function and obtain a useful result so long as the

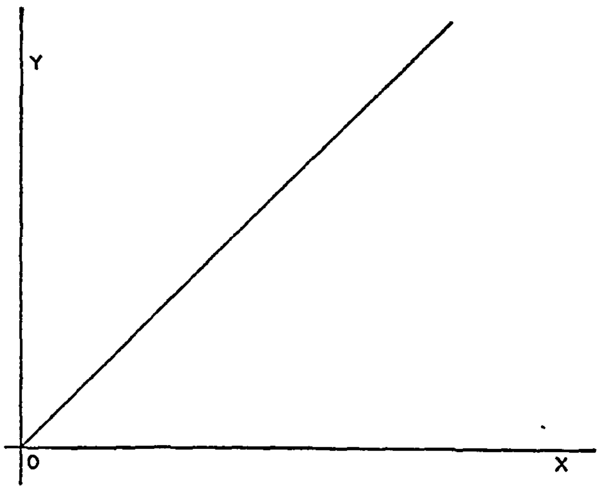

FIG. 7–3

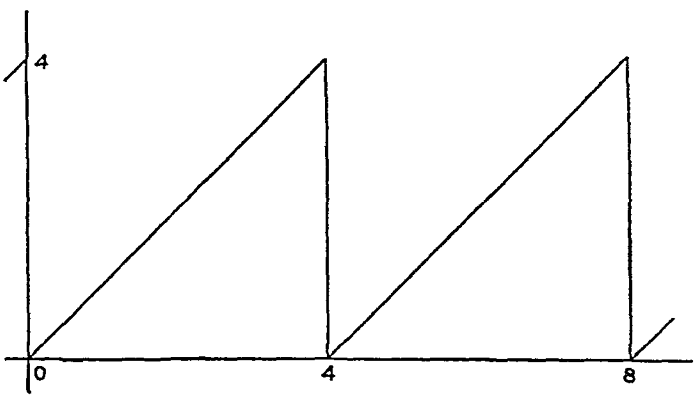

FIG. 7–4

periodic function is identical with the function in Fig. 7–3 in the interval $0 < x < 4$. For example, we can obtain a periodic function as shown in Fig. 7–5. This function is an even function and therefore contains no sine terms. Note that the period is now 8 instead of 4 as it is in Fig. 7–4. The cosine coefficients are obtained from equation (3.36), integrating

through half the period and multiplying the result by 2.

$$a_n = \frac{4}{8}\int_0^4 x \cos\frac{2n\pi x}{8}\,dx \tag{7.50}$$

$$= \frac{1}{2}\left[\frac{x \sin\frac{2n\pi x}{8}}{\frac{2n\pi}{8}} + \frac{\cos\frac{2n\pi x}{8}}{\left(\frac{2n\pi}{8}\right)^2}\right]_0^4 \tag{7.51}$$

$$= \frac{8}{n^2\pi^2}(\cos n\pi - 1) \qquad n \neq 0. \tag{7.52}$$

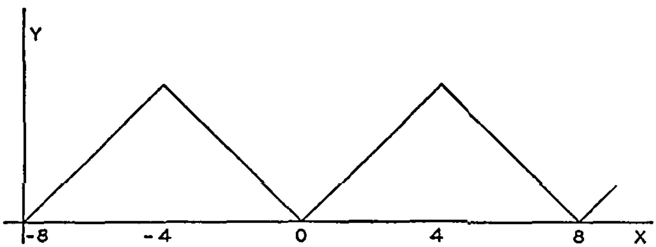

Fig. 7-5

If n is even, $$a_n = 0 \text{ except for } n = 0. \tag{7.53}$$

If n is odd, $$a_n = -\frac{16}{n^2\pi^2}. \tag{7.54}$$

$$a_0 = \frac{4}{8}\int_0^4 x\,dx = 4. \tag{7.55}$$

Therefore, the function $y = x$ in the interval $0 < x < 4$ can be represented by the series

$$y = 2 - \frac{16}{\pi^2}\cos\frac{\pi x}{4} - \frac{16}{9\pi^2}\cos\frac{3\pi x}{4} - \frac{16}{25\pi^2}\cos\frac{5\pi x}{4} - \cdots. \tag{7.56}$$

The problem of expressing the function $y = x$ in the interval $0 < x < 4$ as a sine series is left to the student. The series in equation (7.43) is not a sine series because of the presence of the constant term. There may be a constant term in a cosine series but not in a sine series.

7.10 Differentiation and Integration. In the proof given above for the formulas (7.2) and (7.3), we assumed that the Fourier series could be integrated term by term. The basis for this assumption is found in the

proof of Fourier's theorem. The following discussion gives an indication of the possibilities for repeated integration or differentiation.

The Fourier series, equation (7.1), can represent a function (can converge) only if the two sequences of numbers

$$
\begin{array}{l}
a_1, \quad a_2, \quad a_3, \cdots a_n, \cdots \\
b_1, \quad b_2, \quad b_3, \cdots b_n, \cdots
\end{array}
$$

approach zero fast enough. If we integrate the original series term by term the corresponding sequences for the resulting series will be

$$
\begin{array}{l}
a_1, \quad \dfrac{a_2}{2}, \quad \dfrac{a_3}{3}, \cdots \dfrac{a_n}{n}, \cdots \\[2ex]
b_1, \quad \dfrac{b_2}{2}, \quad \dfrac{b_3}{3}, \cdots \dfrac{b_n}{n}, \cdots
\end{array}
$$

which approach zero faster than for the given series; therefore we can integrate as many times as we wish.

Now if we differentiate the given series term by term the corresponding sequences for the new series will be

$$
\begin{array}{l}
a_1, \quad 2a_2, \quad 3a_3, \cdots na_n, \cdots \\
b_1, \quad 2b_2, \quad 3b_3, \cdots nb_n, \cdots
\end{array}
$$

which do not approach zero as fast as the sequences for the original series. The differentiated series may or may not converge, depending upon the particular function. It is interesting to note here that the series obtained by differentiating equation (7.43) does not converge while the series obtained by differentiating equation (7.56) does converge, although both (7.43) and (7.56) represent the same function in the interval $0 < x < 4$.

7.11 Approximate Fourier Analysis. A curve to be represented by a Fourier series may be obtained on an oscillograph and in this case we do not have any expression for $f(x)$. If we do not have an analytical expression for $f(x)$ to substitute in formulas (7.3) and (7.4), or if we are unable to perform the indicated integration, we must have some way of approximating the integrals.

Assume a given curve is to be analyzed to find a_1. Each ordinate of the given curve is multiplied by $\cos x$ to give a new curve. The area under this new curve from $x = 0$ to $x = 2\pi$ is to be divided by π to give a_1. The interval from 0 to 2π is cut into k intervals each Δx in length; $k\Delta x = 2\pi$. If ordinates are erected at each point of division of the interval and, if we then connect successive points of intersection of the curve and these ordinates by straight lines, the curve will be approximated by a series of k straight lines. See Fig. 7-6. The area will be approximated by the sum of the areas of k trapezoids.

The area of the first trapezoid is

$$\Delta x \left(\frac{y_0 + y_1}{2}\right).$$

The area of the next trapezoid is

$$\Delta x \left(\frac{y_1 + y_2}{2}\right).$$

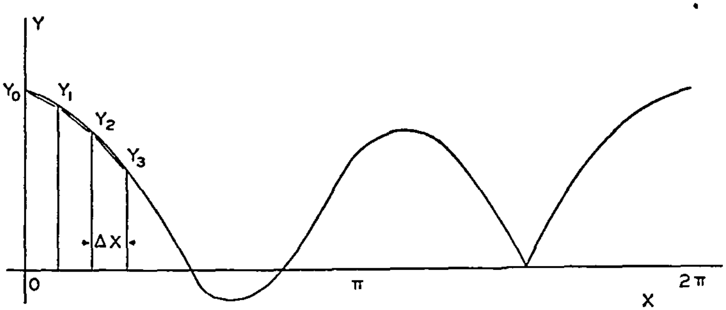

FIG. 7–6

The total area is, therefore,

$$\text{area} = \Delta x \left[\frac{y_0}{2} + y_1 + y_2 + y_3 + \cdots + \frac{y_k}{2}\right]. \tag{7.57}$$

The desired coefficient a_1 is obtained by dividing the area above by π

$$a_1 = \frac{\text{area}}{\pi} = \frac{2(\text{area})}{k\Delta x}. \tag{7.58}$$

If we now substitute the formal expressions for the ordinates, we obtain the formula for a_1.

$$a_1 = \frac{2}{k}\left[\frac{f(0)}{2} + f(\Delta x) \cos \Delta x + f(2\Delta x) \cos 2\Delta x + \cdots + \frac{f(2\pi)}{2}\right],$$

$$a_1 = \frac{1}{k}\,[f(0) + f(2\pi)] + \frac{2}{k} \sum_{u=1}^{u=k-1} f(u\Delta x) \cos u\Delta x. \tag{7.59}$$

In the special case where $f(0) = f(2\pi)$ this becomes

$$a_1 = \frac{2}{k} \sum_{u=1}^{u=k} f(u\Delta x) \cos u\Delta x, \qquad \text{where } f(0) = f(2\pi). \tag{7.60}$$

The formula is often found in this form and one must be sure that it fits the problem on hand.

The approximate formulas for a_n and b_n are given without proof as

$$a_n = \frac{1}{k}[f(0) + f(2\pi)] + \frac{2}{k}\sum_{u=1}^{u=k-1} f(u\Delta x) \cos nu\Delta x, \tag{7.61}$$

$$b_n = \frac{2}{k}\sum_{u=1}^{u=k} f(u\Delta x) \sin nu\Delta x. \tag{7.62}$$

In using these formulas it is desirable for the sake of accuracy to follow some standard form of table. One recommended is shown in Table VII–1 where an approximation to a_2 is carried out.

Example. An oscillogram of a periodic function has been made. The oscillogram has been measured so that the value of the function is known at twenty-four equally spaced points during the period. These values are listed below. This corresponds to 15° intervals where the period is 360°.

$f(0) = 3.11$	$f(135°) = 8.18$	$f(255°) = 1.24$
$f(15°) = 5.12$	$f(150°) = 7.91$	$f(270°) = -0.82$
$f(30°) = 7.24$	$f(165°) = 7.49$	$f(285°) = -3.22$
$f(45°) = 8.32$	$f(180°) = 6.91$	$f(300°) = -4.84$
$f(60°) = 8.78$	$f(195°) = 6.24$	$f(315°) = -4.01$
$f(75°) = 9.19$	$f(210°) = 5.51$	$f(330°) = 0.44$
$f(90°) = 9.41$	$f(225°) = 4.32$	$f(345°) = 1.23$
$f(105°) = 9.21$	$f(240°) = 3.11$	$f(360°) = 3.11$
$f(120°) = 8.65$		

The values of the function are entered in the table and the indicated steps carried out. If the product $f(u\Delta x) \cos 2u\Delta x$ is positive it is entered in the column headed "positive." If the product is negative it is entered in the column headed "negative." Finally the sum of the positive column minus the sum of the negative column is divided by 12 to obtain a_2.

7.12 Results of Approximate Analyses. When an approximate Fourier analysis is to be made there are two questions of importance that come up. How many measurements should be made on the curve, and how close will the approximations be to the correct values of the coefficients? The answers to these questions of course depend on the function being analyzed; nevertheless, the following discussion of the approximate analysis will be of considerable help in aiding one to plan the analysis of a particular curve.

If we make twenty-four measurements on the curve to be analyzed we would expect to be able to obtain twenty-four coefficients, certainly no more than twenty-four. This means that, although we can substitute any value of n in formulas (7.61) and (7.62) to compute the various values of a_n and b_n, we get no more than twenty-four different numerical results. It will be seen below that for k equal to twenty-four we do get twenty-four different coefficients, a_0 to a_{12} and b_1 to b_{11}. We shall also see that the trapezoidal rule as worked out in the preceding section is better than Simpson's rule, which is rather surprising.

TABLE VII–1

Cosine coefficient of second harmonic

u	$u\Delta x$	$f(u\Delta x)$	$2u\Delta x$	$\cos 2u\Delta x$	$f(u\Delta x) \cos 2u\Delta x$	
					Positive	Negative
0	0	3 11	0	1 000	3 11	
1	15°	5 12	30°	0 866	4 43	
2	30°	7 24	60°	0 500	3 62	
3	45°	8 32	90°	0 000		
4	60°	8 78	120°	−0 500		4 39
5	75°	9 19	150°	−0 866		7 96
6	90°	9 41	180°	−1 000		9 41
7	105°	9 21	210°	−0 866		7.98
8	120°	8 65	240°	−0 500		4.33
9	135°	8 18	270°	0 000		
10	150°	7 91	300°	0 500	3 95	
11	165°	7 49	330°	0 866	6 49	
12	180°	6 91	360°	1 000	6 91	
13	195°	6 24	390°	0 866	5 40	
14	210°	5 51	420°	0 500	2 75	
15	225°	4 32	450°	0 000		
16	240°	3 11	480°	−0 500		1 55
17	255°	1 24	510°	−0 866		1.07
18	270°	−0 82	540°	−1 000	0 82	
19	285°	−3 22	570°	−0 866	2 79	
20	300°	−4 84	600°	−0 500	2 42	
21	315°	−4 01	630°	0 000		
22	330°	0 44	660°	0 500	0.22	
23	345°	1 23	690°	0 866	1 07	
24	360°	3 11	720°	1 000	Note	
					43 98	36 69
					36 69	
					7.29 ÷ 12 = 0 607 = a_2	

Note· If $f(0) = f(360°)$ enter only one
If $f(0) \neq f(360°)$ enter half of each.

Let a_n' be the approximation to a_n obtained by using formula (7.60). Let b_n' be the approximation to b_n obtained by using formula (7.62).

$$a_n' = \frac{2}{k} \sum_{u=1}^{k} f(u\Delta x) \cos nu\Delta x. \tag{7.63}$$

$$b_n' = \frac{2}{k} \sum_{u=1}^{k} f(u\Delta x) \sin nu\Delta x. \tag{7.64}$$

The following device enables us to obtain the desired results very quickly. Define C_n' as

$$C_n' = a_n' + ib_n'. \tag{7.65}$$

Substitute the values of a'_n and b'_n into (7.65) and we have

$$C'_n = \frac{2}{k}\sum_{u=1}^{k} f(u\Delta x)\cos nu\Delta x + i\frac{2}{k}\sum_{u=1}^{k} f(u\Delta x)\sin nu\Delta x \tag{7.66}$$

$$= \frac{2}{k}\sum_{u=1}^{k} f(u\Delta x)\ (\cos nu\Delta x + i\sin nu\Delta x) \tag{7.67}$$

$$= \frac{2}{k}\sum_{u=1}^{k} f(u\Delta x)e^{inu\Delta x}. \tag{7.68}$$

We now substitute the Fourier series for $f(u\Delta x)$ in (7.68). The series is

$$f(u\Delta x) = 0.5a_0 + \sum_{m=1}^{\infty} a_m \cos mu\Delta x + \sum_{m=1}^{\infty} b_m \sin mu\Delta x. \tag{7.69}$$

This gives us for (7.68)

$$C'_n = \frac{2}{k}\sum_{u=1}^{k}\left[0.5a_0 + \sum_{m=1}^{\infty} a_m\cos mu\Delta x + \sum_{m=1}^{\infty} b_m \sin mu\Delta x\right]e^{inu\Delta x} \tag{7.70}$$

$$= \frac{a_0}{k}\sum_{u=1}^{k} e^{inu\Delta x} + \frac{2}{k}\sum_{u=1}^{k}\sum_{m=1}^{\infty} a_m e^{inu\Delta x}\cos mu\Delta x$$

$$+ \frac{2}{k}\sum_{u=1}^{k}\sum_{m=1}^{\infty} b_m e^{inu\Delta x}\sin mu\Delta x, \tag{7.71}$$

$$C'_n = \frac{a_0}{k}\sum_{u=1}^{k} e^{inu\Delta x} + \frac{2}{k}\sum_{m=1}^{\infty}\sum_{u=1}^{k} a_m e^{inu\Delta x}\cos mu\Delta x$$

$$+ \frac{2}{k}\sum_{m=1}^{\infty}\sum_{u=1}^{k} b_m e^{inu\Delta x}\sin mu\Delta x, \tag{7.72}$$

$$C'_n = a_{n0} + \sum_{m=1}^{\infty} a_{nm} + \sum_{m=1}^{\infty} b_{nm} \tag{7.73}$$

where

$$a_{n0} = \frac{a_0}{k}\sum_{u=1}^{k} e^{inu\Delta x} \tag{7.74}$$

is the contribution a_0 makes to C'_n, and

$$a_{nm} = \frac{2a_m}{k}\sum_{u=1}^{k} e^{inu\Delta x}\cos mu\Delta x \tag{7.75}$$

is the contribution a_m makes to C'_n.

$$b_{nm} = \frac{2b_m}{k}\sum_{u=1}^{k} e^{inu\Delta x}\sin mu\Delta x \tag{7.76}$$

is the contribution b_m makes to C'_n. The analysis will be correct if

$$a_{nm} = 0 \text{ for } n \neq m, \qquad b_{nm} = 0 \text{ for } n \neq m,$$
$$a_{nn} = a_n, \qquad b_{nn} = ib_n.$$

We shall now determine the value of a_{n0} from equation (7.74). Note that the sum in equation (7.74) is the sum of a geometric progression. The formula for the sum of such a progression is

$$S = \frac{a(r^k - 1)}{r - 1} \tag{7.77}$$

where S is the sum, a is the first term, r is the ratio of one term to the term that precedes it, and k is the number of terms. In this case

$$a = e^{in\Delta x},$$
$$r = e^{in\Delta x},$$
$$k = k.$$

Therefore, we have for a_{n0}

$$a_{n0} = \frac{a_0}{k}\frac{e^{in\Delta x}(e^{ink\Delta x} - 1)}{e^{in\Delta x} - 1}. \tag{7.78}$$

Now $k\Delta x = 2\pi$. Therefore the numerator of the above fraction is always zero. If $n\Delta x$ is not a multiple of 2π the denominator is not zero and therefore the fraction is zero. If $n\Delta x$ is not a multiple of 2π, since $k\Delta x = 2\pi$, n is not a multiple of k. Therefore, we have as a result

$$a_{n0} = 0, \quad \text{if } n \text{ is not a multiple of } k. \tag{7.79}$$

If n is a multiple of k, $n\Delta x$ is a multiple of 2π and the first term, a_0 of the progression is unity and also the ratio r is equal to unity. Therefore, we have the sum of k terms each equal to unity and equation (7.74) becomes

$$a_{n0} = a_0, \quad \text{if } n \text{ is a multiple of } k. \tag{7.80}$$

We are now ready to investigate a_{nm}

$$a_{nm} = \frac{2a_m}{k}\sum_{u=1}^{k} e^{inu\Delta x}\cos mu\Delta x, \tag{7.75}$$

$$a_{nm} = \frac{2a_m}{k}\sum_{u=1}^{k}\frac{e^{imu\Delta x} + e^{-imu\Delta x}}{2}e^{inu\Delta x} \tag{7.81}$$

$$= \frac{a_m}{k}\sum_{u=1}^{k}[e^{i(m+n)u\Delta x} + e^{i(n-m)u\Delta x}]. \tag{7.82}$$

The right-hand side of equation (7.82) contains two geometric progressions

that can be evaluated as was done above.

$$\sum_{u=1}^{k} e^{i(m+n)u\Delta x} = k, \quad \text{if } m + n \text{ is a multiple of } k,$$

$$\sum_{u=1}^{k} e^{i(m+n)u\Delta x} = 0, \quad \text{if } m + n \text{ is not a multiple of } k,$$

$$\sum_{u=1}^{k} e^{i(n-m)u\Delta x} = k, \quad \text{if } n - m \text{ is a multiple of } k,$$

$$\sum_{u=1}^{k} e^{i(n-m)u\Delta x} = 0, \quad \text{if } n - m \text{ is not a multiple of } k.$$

We therefore have for a_{nm}

$$a_{nm} = a_m, \text{ if } m + n \text{ is a multiple of } k \text{ and } m - n \text{ is not, or if } m - n \text{ is a multiple of } k \text{ and } m + n \text{ is not.} \tag{7.83}$$

$$a_{nm} = 2a_m, \text{ if both } m + n \text{ and } m - n \text{ are multiples of } k, \tag{7.84}$$

$$a_{nm} = 0, \text{ if neither } m + n \text{ nor } m - n \text{ are multiples of } k. \tag{7.85}$$

We can remark here that since a_{nm} is always real the cosine coefficients do not produce any error in the approximations for the sine coefficients.

Let us now evaluate b_{nm}.

$$b_{nm} = \frac{2b_m}{k} \sum_{u=1}^{k} e^{inu\Delta x} \sin mu\Delta x \tag{7.76}$$

$$= \frac{2b_m}{k} \sum_{u=1}^{k} \frac{e^{imu\Delta x} - e^{-imu\Delta x}}{i2} e^{inu\Delta x} \tag{7.86}$$

$$= \frac{ib_m}{k} \sum_{u=1}^{k} [e^{i(n-m)u\Delta x} - e^{i(n+m)u\Delta x}]. \tag{7.87}$$

The geometric progressions on the right-hand side of equation (7.87) are the same as those on the right-hand side of equation (7.82). We can now list the possible values for b_{nm}.

$$b_{nm} = ib_m, \text{ if } n - m \text{ is a multiple of } k \text{ and } m + n \text{ is not,} \tag{7.88}$$

$$b_{nm} = -ib_m, \text{ if } m + n \text{ is a multiple of } k \text{ and } n - m \text{ is not,} \tag{7.89}$$

$$b_{nm} = 0, \text{ if both } n - m \text{ and } m + n \text{ are multiples of } k, \text{ or if neither } n - m \text{ nor } m + n \text{ are multiples of } k. \tag{7.90}$$

These results will be more readily understood when we consider some numerical examples. Suppose measurements are made every 15° so that $k = 24$. Then the approximate value of a_0 will be

$$a_0' = a_0 + 2a_{24} + 2a_{48} + \cdots. \tag{7.91}$$

The approximate value of a_{24} will be the same, in fact,

$$a_0' = a_{24}' = a_{48}' = \cdots. \tag{7.92}$$

The subscripts in equations (7.91) and (7.92) are the same. The approximate value of b_{24} will be zero.

$$b_{24}' = b_{48}' = \cdots = 0, \tag{7.93}$$

The approximate value for a_1 will be

$$a_1' = a_1 + a_{23} + a_{25} + a_{47} + a_{49} + \cdots \tag{7.94}$$

and we also have the equality

$$a_1' = a_{23}' = a_{25}' = a_{47}' = a_{49}' = \cdots. \tag{7.95}$$

The approximate value of b_1 is

$$b_1' = b_1 - b_{23} + b_{25} - b_{47} + b_{49} - \cdots \tag{7.96}$$

and we have the equation

$$b_1' = -b_{23}' = b_{25}' = -b_{47}' = \cdots. \qquad (7.96)$$

The approximate value of b_3 will be

$$b_3' = b_3 - b_{21} + b_{27} - b_{45} + \cdots \tag{7.98}$$

$$b_3' = -b_{21}' = b_{27}' = -b_{45}' = \cdots. \tag{7.99}$$

These results can be visualized with the aid of Table VII–2

TABLE VII–2

Approximate coefficients for $k = 24$

Cosines: add								
0	24	24	48	48	72	72	96	96
1	23	25	47	49	71	73	95	97
2	22	26	46	50	70	74	94	98
3	21	27	45	51	69	75	93	99
4	20	28	44	52	68	76	92	100
5	19	29	43	53	67	77	91	101
6	18	30	42	54	66	78	90	102
7	17	31	41	55	65	79	89	103
8	16	32	40	56	64	80	88	104
9	15	33	39	57	63	81	87	105
10	14	34	38	58	62	82	86	106
11	13	35	37	59	61	83	85	107
12	12	36	36	60	60	84	84	108
Sines: alternate sign								

The figures in Table VII–2 are the subscripts, read across any horizontal line.

$$a_6' = a_6 + a_{18} + a_{30} + a_{42} + \cdots \tag{7.100}$$

$$a_6' = a_{18}' = a_{30}' = a_{42}' = \cdots \tag{7.101}$$

$$b_6' = b_6 - b_{18} + b_{30} - b_{42} + \cdots \tag{7.102}$$

$$b_6' = -b_{18}' = b_{30}' = -b_{42}' = \cdots \tag{7.103}$$

$$a_{12}' = 2a_{12} + 2a_{36} + 2a_{60} + \cdots \tag{7.104}$$

$$b_{12}' = 0 = b_{36}' = b_{60}' = \cdots \tag{7.105}$$

$$a_{11}' = a_{11} + a_{13} + a_{35} + a_{37} + \cdots \tag{7.106}$$

$$b_{11}' = b_{11} - b_{13} + b_{35} - b_{37} + \cdots \tag{7.107}$$

Now we would seldom expect a_{13} to be negligible compared to a_{11}, and we would hardly expect b_{13} to be negligible when compared with b_{11}. Therefore, if the eleventh harmonic is desired, we must use more than twenty-four measurements on the curve.

On the other hand, we might well expect a_{18} to be negligible compared with a_6 if the given function has no sharp peaks. And therefore a_6' would be a good approximation to a_6 when $k = 24$. If a_7' and a_8' are small compared with a_6' this would give further assurance. It must be recognized that we can make no general rules; we can only make suggestions that we expect will work most of the time.

The results may be summed up as follows: Use at least four times as many measurements as the order of the highest harmonic required. Evaluate several approximate coefficients for higher harmonics to be assured the series converges rapidly enough. Try both even and odd harmonics because even and odd harmonic coefficients may converge at different rates. Remember that these are only guides that may be expected to apply to most common engineering curves. It is always possible that we may need more than the number of measurements recommended.

7.13 Simpson's Rule. It is common practice in order to increase the accuracy to approximate a curve with a set of parabolas rather than the set of straight lines as is done with the trapezoidal rule. When the area under the parabolas is used as an approximation for the area under the curve we are using Simpson's rule.

Let a_n'' be the approximation to a_n when Simpson's rule is used. We may write a_n'' as follows:

$$\begin{aligned} a_n'' = \frac{2}{3k}[f(0)\cos 0 &+ 4f(\Delta x)\cos n\Delta x + 2f(2\Delta x)\cos 2n\Delta x \\ &+ 4f(3\Delta x)\cos 3n\Delta x + 2f(4\Delta x)\cos 4n\Delta x \\ &+ \cdots + f(2\pi)\cos 2\pi]. \end{aligned} \tag{7.108}$$

Assuming that $f(0) = f(2\pi)$, we may write this

$$a_n'' = \frac{8}{3k}\sum_{u=1}^{k} f(u\Delta x)\cos nu\Delta x - \frac{4}{3k}\sum_{u=1}^{0.5k} f(2u\Delta x)\cos 2nu\Delta x, \qquad (7.109)$$

$$a_n'' = \tfrac{4}{3}a_n' - \tfrac{1}{3}A_n' \qquad (7.110)$$

where a_n' is the approximation to a_n when the trapezoidal rule is used, and the same number of measurements is employed as was used in Simpson's rule. The term A_n' is also an approximation obtained with the trapezoidal rule but using half the number of measurements as was used for Simpson's rule. In order to know what we have for $k = 24$, we need a table similar to Table VII-2, but for $k = 12$. Such a table is included; see Table VII-3.

TABLE VII-3

Approximate coefficients for $k = 12$

Cosines: add								
0	12	12	24	24	36	36	48	48
1	11	13	23	25	35	37	47	49
2	10	14	22	26	34	38	46	50
3	9	15	21	27	33	39	45	51
4	8	16	20	28	32	40	44	52
5	7	17	19	29	31	41	43	53
6	6	18	18	30	30	42	42	54
Sines: alternate sign								

The approximate value for a_3 obtained when Simpson's rule is used, where $k = 24$, is

$$\begin{aligned} a_3'' = {} & \tfrac{4}{3}[a_3 + a_{21} + a_{27} + a_{45} + \cdots] \\ & -\tfrac{1}{3}[a_3 + a_9 + a_{15} + a_{21} + a_{27} + a_{33} + a_{39} + a_{45} + \cdots], \end{aligned} \qquad (7.111)$$

$$\begin{aligned} a_3'' = {} & a_3 + a_{21} + a_{27} + a_{45} + \cdots - \tfrac{1}{3}a_9 - \tfrac{1}{3}a_{15} \\ & - \tfrac{1}{3}a_{33} - \tfrac{1}{3}a_{39} - \cdots, \end{aligned} \qquad (7.112)$$

while the trapezoidal rule gives for a_3, when $k = 24$,

$$a_3' = a_3 + a_{21} + a_{27} + a_{45} + \cdots. \qquad (7.113)$$

Now the lowest order harmonic to cause error for a_3' is a_{21}, while for a_3'' it is a_9. We would generally expect a_3' to be nearer the correct value a_3 than is a_3'', and the simpler trapezoidal rule is to be preferred to Simpson's rule when making an approximate Fourier analysis.

PROBLEMS ON CHAPTER 7

1. Prove that the coefficient of the term $\sin nx$ in the Fourier series is given by the formula.

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx.$$

2. Find the sine and cosine coefficients for the fundamental, b_1 and a_1, for the function shown in Fig. 7–7.

3. Find the sine coefficients for the fundamental and second harmonic, b_1 and b_2, for the function shown in Fig. 7–8.

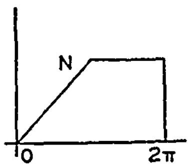

FIG. 7–7

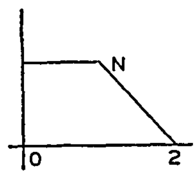

FIG. 7–8

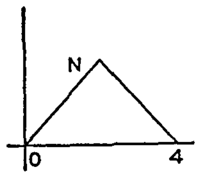

FIG. 7–9

4. Find the cosine coefficient for the fifth harmonic, a_5, for the function shown in Fig. 7–11.

5. Express the function shown in Fig. 7–7 as a Fourier series.

6. Express the function shown in Fig. 7–8 as a Fourier series.

7. Express the function shown in Fig. 7–9 as a Fourier series.

8. Express the function shown in Fig. 7–10 as a Fourier series.

9. Express the function shown in Fig. 7–11 as a Fourier series.

10. Express the function shown in Fig. 7–12 as a Fourier series.

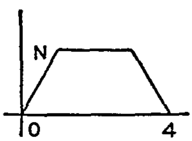

FIG. 7–10

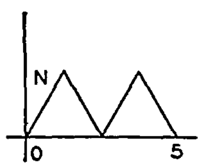

FIG. 7–11

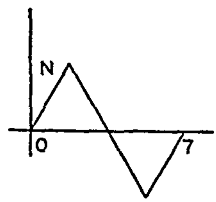

FIG. 7–12

11. Change the series obtained in problem 5 to one containing only cosines.
12. Change the series obtained in problem 5 to one containing only sines.
13. Change the series obtained in problem 6 to one containing only cosines.
14. Change the series obtained in problem 6 to one containing only sines.
15. Change the series obtained in problem 7 to one containing only sines.
16. Change the series obtained in problem 7 to one containing only cosines.
17. Change the series obtained in problem 8 to one containing only sines.
18. Change the series obtained in problem 8 to one containing only cosines.
19. Change the series obtained in problem 9 to one containing only cosines.
20. Change the series obtained in problem 9 to one containing only sines.

21. Change the series obtained in problem 10 to one containing only cosines.
22. Change the series obtained in problem 10 to one containing only sines.
23. Show that an odd function can be expressed only as the sum of odd functions.
24. Show that an even function can be expressed only as the sum of even functions.
25. Express the function shown in Fig. 7–7 as a sine series valid for the interval shown.
26. Express the function shown in Fig. 7–7 as a cosine series valid for the interval shown.
27. Express the function shown in Fig. 7–8 as a sine series valid for the interval shown.
28. Express the function shown in Fig. 7–8 as a cosine series valid for the interval shown.
29. Express the function shown in Fig. 7–9 as a sine series valid in the interval shown.
30. Express the function shown in Fig. 7–9 as a cosine series valid in the interval shown.
31. Express the function shown in Fig. 7–10 as a sine series valid in the interval shown.
32. Express the function shown in Fig. 7–10 as a cosine series valid in the interval shown.
33. Express the function shown in Fig. 7–11 as a sine series valid in the interval shown.
34. Express the function shown in Fig. 7–11 as a cosine series valid in the interval shown.
35. Express the function shown in Fig. 7–12 as a sine series valid in the interval shown.
36. Express the function shown in Fig. 7–12 as a cosine series valid in the interval shown.

In the following problems use the data on the oscillogram given in section 7.11. Make suitable tables to carry out the computations.

37. Determine the constant term $0.5a_0$ and the sine coefficient of the second harmonic, b_2.
38. Determine the coefficients of the fundamental, a_1 and b_1.
39. Determine the coefficients of the third harmonic, a_3 and b_3.
40. Determine the coefficients of the fourth harmonic, a_4 and b_4.

CHAPTER 8

DIFFERENTIAL EQUATIONS

8.1 Engineering Problems. Many problems that face the engineer lead to equations involving numerical relations among several unknowns. These equations might include powers, roots, cosines, logarithms, etc. In each case the information in the physical problem is translated completely into one or more equations which are solvable by standard algebraic methods. The process can be illustrated by the chart in Fig. 8–1.

On the other hand, many problems dealing with rates, such as velocities, accelerations, and slopes, lead to equations which involve derivatives as well as simple functions. Such equations are called differential equations. It happens that in each case the differential equations are set up before all the information in the problem is exhausted. Therefore it is impossible to obtain a unique solution from the differential equations. The solution that is obtained is called the general solution of the differential equations. To obtain the particular solution for the physical problem from the general solution we must use that part of the physical problem not already used to set up the differential equations. This part of the physical problem is known as the boundary conditions. The procedure is illustrated in the chart of Fig. 8–2.

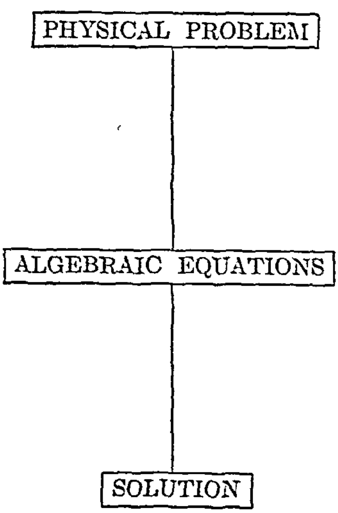

FIG. 8–1

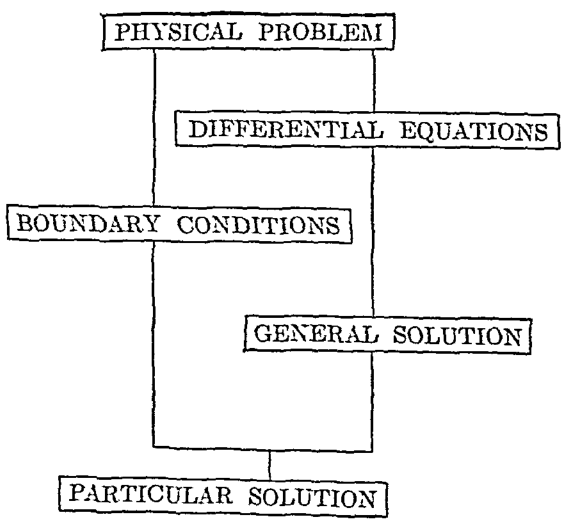

FIG. 8–2

8.2 Development of Differential Equation from its Solution. In making a study of algebraic equations, we obtain valuable information by considering how a polynomial equation can be derived from a knowledge of its roots. We find that the corresponding study in the case of differential equations is equally helpful. It can be shown by substitution that

$$q = K\,e^{-t/RC} \tag{8.1}$$

is a solution of the differential equation

$$R\frac{dq}{dt} + \frac{q}{C} = 0. \tag{8.2}$$

We obtain different functions by assigning different values to K. The differential equation is satisfied by any of the family of functions obtained by setting K equal to various values. We therefore say that the differential equation is the equation of the family of curves. Given the equation

$$q = K\,e^{-t/RC}$$

we can obtain the differential equation if we solve for K and then differentiate

$$K = q\,e^{t/RC}, \tag{8.3}$$

$$0 = \frac{q}{RC}e^{t/RC} + \frac{dq}{dt}e^{t/RC}. \tag{8.4}$$

This can be rearranged and after multiplying by

$$e^{-t/RC}$$

we have

$$R\frac{dq}{dt} + \frac{q}{C} = 0. \tag{8.5}$$

Another way of obtaining the differential equation from the relation

$$q = K\,e^{-t/RC}$$

is to differentiate and get

$$\frac{dq}{dt} = -\frac{K}{RC}e^{-t/RC}$$

and eliminate K from these two equations.

As another example consider the case of all parabolas tangent to the x axis and having their axes parallel to the y axis. The equation of such a

parabola is

$$y = p(x - h)^2$$

where the values of p and h determine the particular parabola. To obtain the differential equation we can solve for h and differentiate to eliminate h.

$$h = x - \sqrt{\frac{y}{p}}, \tag{8.6}$$

$$0 = 1 - \frac{1}{2\sqrt{py}}\frac{dy}{dx}. \tag{8.7}$$

Now solve for $\sqrt{p}$ and differentiate to eliminate p.

$$\sqrt{p} = \frac{1}{2\sqrt{y}}\frac{dy}{dx}, \tag{8.8}$$

$$0 = -\frac{1}{4\sqrt{y^3}}\left(\frac{dy}{dx}\right)^2 + \frac{1}{2\sqrt{y}}\frac{d^2y}{dx^2}, \tag{8.9}$$

$$2y\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \tag{8.10}$$

is the desired equation. Using the second method suggested we differentiate the given expression twice and get three equations

$$y = p(x - h)^2, \tag{8.11}$$

$$\frac{dy}{dx} = 2p(x - h), \tag{8.12}$$

$$\frac{d^2y}{dx^2} = 2p. \tag{8.13}$$

Eliminate $(x - h)$ from the first two equations which gives

$$\frac{1}{4p^2}\left(\frac{dy}{dx}\right)^2 = \frac{y}{p},$$

$$\left(\frac{dy}{dx}\right)^2 = 4py.$$

This equation with (8.13) gives the desired result

$$\left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}. \tag{8.10}$$

The important thing to notice in these examples is that we have to differentiate once for each constant that is being eliminated. We there-

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$$\frac{dq}{dt} = -\frac{K}{RC}e^{-t/RC}$$

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As another example consider the case of all parabolas tangent to the x axis and having their axes parallel to the y axis. The equation of such a

parabola is

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where the values of p and h determine the particular parabola. To obtain the differential equation we can solve for h and differentiate to eliminate h.

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Now solve for $\sqrt{p}$ and differentiate to eliminate p.

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$$\frac{dy}{dx} = 2p(x - h), \tag{8.12}$$

$$\frac{d^2y}{dx^2} = 2p. \tag{8.13}$$

Eliminate $(x - h)$ from the first two equations which gives

$$\frac{1}{4p^2}\left(\frac{dy}{dx}\right)^2 = \frac{y}{p},$$

$$\left(\frac{dy}{dx}\right)^2 = 4py.$$

This equation with (8.13) gives the desired result

$$\left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}. \tag{8.10}$$

The important thing to notice in these examples is that we have to differentiate once for each constant that is being eliminated. We there-

fore assume that a differential equation containing a fifth derivative, but no higher derivative, must come from an expression containing five constants. In other words the solution of a differential equation should have as many arbitrary constants as the order of the highest order derivative.

8.3 Definitions. An equation containing derivatives is a differential equation. If the equation contains partial derivatives it is a partial differential equation. If it contains only ordinary derivatives it is an ordinary differential equation.

The order of the highest order derivative is the order of the equation.

The degree of the highest order derivative is the degree of the equation. Both of the following equations are second order equations of the first degree.

$$\frac{d^2y}{dx^2} + 3y = x.$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0.$$

The following equation is a third order equation of the seventh degree.

$$\left(\frac{d^3y}{dx^3}\right)^7 + 2\left(\frac{dy}{dx}\right)^9 + 5 \sin x = 0.$$

Any expression such as $y = f(x)$ which is found to satisfy a differential equation is called a solution of the equation. If the expression $y = f(x)$ contains as many arbitrary constants as the order of the equation, $y = f(x)$ is called the general solution, otherwise $y = f(x)$ is a particular solution or a particular integral. In the first example above

$$q = K\, e^{-t/RC}$$

is the general solution of

$$R\frac{dq}{dt} + \frac{q}{C} = 0$$

while a particular solution to the same equation is

$$q = 35\, e^{-t/RC}.$$

The general solution of the equation

$$\left(\frac{dy}{dx}\right)^2 = 2y\frac{d^2y}{dx^2}$$

is

$$y = p(x - h)^2.$$

The equations

$$y = x^2, \quad y = 5x^2, \quad \text{and} \quad y = 2(x - h)^2$$

are all particular solutions.

8.4 Variables Separable. Any first order differential equation of the first degree can be written in the form

$$\frac{dy}{dx} = F(x, y). \tag{8.14}$$

If it happens that $F(x, y)$ is such a function of x and y that the equation can be written

$$F_1(y)\, dy + F_2(x)\, dx = 0 \tag{8.15}$$

we say the variables are separable; in fact, in equation (8.15) they are separated, and we can integrate, obtaining

$$\int F_1(y)\, dy + \int F_2(x)\, dx = C \tag{8.16}$$

where C is the constant of integration required by a first order differential equation. As an example consider the equation

$$\frac{dy}{dx} = x^2 y. \tag{8.17}$$

This can be written

$$\frac{dy}{y} - x^2\, dx = 0 \tag{8.18}$$

and integrated to

$$\overset{\text{Log}}{\ln}\, y - \frac{x^3}{3} = C. \tag{8.19}$$

Solving for y in terms of x gives as a solution

$$y = e^{C + x^3/3}. \tag{8.20}$$

To check this result differentiate equation (8.20)

$$\frac{dy}{dx} = x^2\, e^{C + x^3/3} \tag{8.21}$$

$$= x^2 y \tag{8.22}$$

which agrees with the given equation.

8.5 Exact Equations. Integration Factors. It sometimes happens that the given equation can be integrated as it stands. In this case it is

said to be an exact equation. The equation

$$\frac{dy}{y} = x^2\,dx$$

is exact because it can be integrated directly giving

$$\ln y = \frac{x^3}{3} + C.$$

The equation

$$\frac{y - x\dfrac{dy}{dx}}{y^2} = 2x \tag{8.23}$$

is an exact equation integrating directly into

$$\frac{x}{y} = x^2 + C. \tag{8.24}$$

In the illustration in the preceding section the equation

$$\frac{dy}{dx} = x^2y$$

was not exact. Multiplying the equation by y^{-1} gives

$$\frac{1}{y}\frac{dy}{dx} = x^2$$

which is exact. The quantity y^{-1} is called an integrating factor.

The equation

$$x\,dy - y\,dx = 3x^3\,dx$$

is made exact by multiplying by the integrating factor x^{-2} giving

$$\frac{x\,dy - y\,dx}{x^2} = 3x\,dx \tag{8.25}$$

which integrates into

$$\frac{y}{x} = \frac{3x^2}{2} + C. \tag{8.26}$$

The integrals listed in Table VIII–1 are often useful as a guide to finding integrating factors. Use of an integrating factor is made in solving linear first order equations in the following section.

TABLE VIII–1

$$\int (x\,dy + y\,dx) = xy + C$$

$$\int \frac{x\,dy - y\,dx}{x^2} = \frac{y}{x} + C$$

$$\int \frac{x\,dy - y\,dx}{y^2} = -\frac{x}{y} + C$$

$$\int \frac{x\,dy + y\,dx}{xy} = \ln(xy) + C$$

$$\int \frac{x\,dy - y\,dx}{xy} = \ln\left(\frac{y}{x}\right) + C$$

$$\int \frac{x\,dy - y\,dx}{x^2 + y^2} = \arctan\left(\frac{y}{x}\right) + C$$

$$\int \frac{x\,dy - y\,dx}{x^2 - y^2} = \frac{1}{2}\ln\left(\frac{x+y}{x-y}\right) + C$$

8.6 Linear Equations. The equation

$$\frac{dy}{dx} + Py = Q \tag{8.27}$$

where P and Q are functions of x is called a linear equation since no higher power of y or its derivative appears but the first and neither does their product appear. In this case an integrating factor is

$$e^{\int P\,dx}.$$

$$\left(\frac{dy}{dx} + Py\right) e^{\int P\,dx} = Q\,e^{\int P\,dx}. \tag{8.28}$$

The left-hand side of equation (8.28) is the derivative of

$$y\,e^{\int P\,dx}$$

and the right-hand side contains only functions of x. Therefore we have

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx + C. \tag{8.29}$$

Solving for y in terms of x we have

$$y = e^{-\int P\,dx}\int Q\,e^{\int P\,dx}\,dx + C\,e^{-\int P\,dx} \tag{8.30}$$

As an illustration consider the equation

$$\frac{dy}{dx} + 3y = 2.$$

$$\frac{f(v, 1)\,dv}{vf(v, 1) - 1} + \frac{dy}{y} = 0. \tag{8.43}$$

As an example consider the equation

$$\frac{dy}{dx} = \frac{2x + y}{x}. \tag{8.44}$$

Substituting $y = vx$, we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = 2 + \frac{y}{x} = 2 + v,$$

$$v + x\frac{dv}{dx} = 2 + v, \tag{8.45}$$

$$v\,dx + x\,dv = 2dx + v\,dx,$$

$$dv = 2\frac{dx}{x}, \tag{8.46}$$

$$v = 2\ln x + C.$$

But $y = vx$, therefore the solution is

$$y = 2x\ln x + Cx. \tag{8.47}$$

The same problem can be worked using the other substitution. We have in this case ($x = vy$).

$$\frac{dx}{dy} = v + y\frac{dv}{dy} = \frac{x}{2x + y} = \frac{\frac{x}{y}}{2\left(\frac{x}{y}\right) + 1} = \frac{v}{2v + 1},$$

$$v + y\frac{dv}{dy} = \frac{v}{2v + 1}, \tag{8.48}$$

$$2v^2\,dy + v\,dy + 2vy\,dv + y\,dv = v\,dy,$$

$$2v^2\,dy + (2v + 1)y\,dv = 0,$$

$$\frac{dy}{y} + \frac{2v + 1}{2v^2}\,dv = 0,$$

$$\frac{dy}{y} + \frac{dv}{v} + \frac{dv}{2v^2} = 0. \tag{8.49}$$

This is now exact and can be integrated giving

$$\ln y + \ln v - \frac{1}{2v} = C, \tag{8.50}$$

$$\ln(yv) - \frac{1}{2v} = C.$$

Since $x = vy$, this gives us

$$\ln x - \frac{y}{2x} = C,$$

$$y = 2x \ln x - 2Cx. \tag{8.51}$$

This solution agrees with that given in equation (8.47) since C is an arbitrary constant.

8.8 Equations Containing One Derivative. Equations of the form

$$\frac{d^n y}{dx^n} = F(x)$$

can be solved by repeated integration. For example, if the given equation is

$$\frac{d^3 y}{dx^3} = 2x^2, \tag{8.52}$$

this can be reduced to a second order equation as follows:

$$\frac{d^2 y}{dx^2} = \int \frac{d^3 y}{dx^3}\, dx = \int 2x^2\, dx = \frac{2x^3}{3} + C_1. \tag{8.53}$$

Continuing this process we have

$$\frac{dy}{dx} = \int \frac{d^2 y}{dx^2}\, dx = \int \frac{2x^3}{3}\, dx + \int C_1\, dx$$

$$= \frac{x^4}{6} + C_1 x + C_2, \tag{8.54}$$

$$y = \int \frac{dy}{dx}\, dx = \int \frac{x^4}{6}\, dx + \int C_1 x\, dx + \int C_2\, dx,$$

$$y = \frac{x^5}{30} + \frac{C_1 x^2}{2} + C_2 x + C_3. \tag{8.55}$$

8.9 Dependent Variable Absent. If no term in the equation contains y, we can substitute

$$p = \frac{dy}{dx}, \qquad \frac{dp}{dx} = \frac{d^2 y}{dx^2}, \quad \text{etc.}$$

In this way we obtain an equation in p and x whose order is one less than the order of the equation in y and x. For example, to solve the equation

$$\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} = 2. \tag{8.56}$$

The above substitution gives

$$\frac{dp}{dx} + 3p = 2. \tag{8.57}$$

This is a first order equation; in fact, it is the same equation that was solved in section 7.8, giving

$$p = \tfrac{2}{3} + C_1 e^{-3x}. \tag{8.58}$$

Therefore, we have now to solve

$$\frac{dy}{dx} = \tfrac{2}{3} + C_1 e^{-3x}.$$

$$y = \frac{2x}{3} - \frac{C_1}{3} e^{-3x} + C_2 \tag{8.59}$$

is the general solution.

8.10 Independent Variable Absent. In equations in which the independent variable x is absent we can take y as the independent variable and p as the dependent variable, where

$$p = \frac{dy}{dx}. \tag{8.60}$$

In this case the higher derivatives of y will not be the same as in the preceding section but are found as follows:

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}, \tag{8.61}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(p\frac{dp}{dy}\right) = \frac{dp}{dy}\frac{dy}{dx}\frac{dp}{dy} + p\frac{d^2p}{dy^2}\frac{dy}{dx}$$

$$= p\left(\frac{dp}{dy}\right)^2 + p^2\frac{d^2p}{dy^2}. \tag{8.62}$$

The resulting equation in p and y is of order one less than the order of the original equation. As an example of this method consider the following equation:

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} = 0. \tag{8.63}$$

Making the substitution indicated above we have

$$p\frac{dp}{dy} + p^2 + 4p = 0. \tag{8.64}$$

This is made linear by dividing by p

$$\frac{dp}{dy} + p = -4. \tag{8.65}$$

Using the method of section 8.8, we have

$$p = -e^{-y}\int 4e^{y}\,dy + C_1 e^{-y} \tag{8.66}$$

$$= -4 + C_1 e^{-y}, \tag{8.67}$$

$$\frac{dy}{dx} = -4 + C_1 e^{-y}. \tag{8.68}$$

In this case the variables are separable.

$$\frac{dy}{-4 + C_1 e^{-y}} = dx. \tag{8.69}$$

This equation can be integrated if we first multiply numerator and denominator of the left-hand side by e^y.

$$\frac{e^y\,dy}{-4e^y + C_1} = dx. \tag{8.70}$$

This integrates directly into

$$-\tfrac{1}{4}\ln(-4e^y + C_1) = x + C_2, \tag{8.71}$$

$$\ln(-4e^y + C_1) = -4x - 4C_2,$$

$$-4e^y + C_1 = e^{-4x-4C_2},$$

$$e^y = \frac{C_1}{4} - \tfrac{1}{4}e^{-4x-4C_2},$$

$$y = \ln\left[\frac{C_1}{4} - \tfrac{1}{4}e^{-4x-4C}\right], \tag{8.72}$$

giving y in terms of x.

8.11 Linear Differential Equations. A differential equation very common in engineering is known as a linear differential equation. This is the type of equation that is frequently encountered in studying mechanical vibrations and electric circuits. A linear differential equation of order n is an equation of the form

$$X_0\frac{d^ny}{dx^n} + X_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + X_{n-1}\frac{dy}{dx} + X_ny = X \tag{8.73}$$

where X_0, X_1, $\cdots$ X_n, and X are all known functions of the independent variable x. If the right-hand side of the equation, X, is zero, the equation is homogeneous; otherwise the equation is said to be complete.

The following equations are homogeneous linear differential equations:

$$\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + y = 0,$$

$$\frac{d^3y}{dx^3} + \sin x\frac{dy}{dx} + y\tan x = 0.$$

The following equations are complete linear differential equations:

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = x,$$

$$\frac{d^3y}{dx^3} + x^3y = \cos x.$$

The solution of the homogeneous linear differential equation is based on the following two theorems.

Theorem I. If $y = f(x)$ is a solution of a homogeneous linear differential equation, $y = kf(x)$ where k is any constant is also a solution. Consider first the following illustration: $y = e^{-x}$ is a solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$$

since we find by substitution that it satisfies the equation

We have

$$e^{-x} + 3(-e^{-x}) + 2e^{-x} = 0.$$

If we substitute $y = 5e^{-x}$, we find

$$5e^{-x} + 3(-5e^{-x}) + 2(5e^{-x}) = 0$$

and the equation is satisfied. Therefore $y = 5e^{-x}$ is also a solution. The proof of the general theorem follows: Given $y = f(x)$ is a solution of the homogeneous linear differential equation

$$X_0\frac{d^ny}{dx^n} + X_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + X_{n-1}\frac{dy}{dx} + X_ny = 0 \tag{8.74}$$

which means that

$$X_0\frac{d^nf(x)}{dx^n} + X_1\frac{d^{n-1}f(x)}{dx^{n-1}} + \cdots + X_{n-1}\frac{df(x)}{dx} + X_nf(x) = 0. \tag{8.75}$$

Substitution of $y = kf(x)$ into the given equation gives

$$X_0 \frac{d^n kf(x)}{dx^n} + X_1 \frac{d^{n-1} kf(x)}{dx^{n-1}} + \cdots + X_{n-1} \frac{dkf(x)}{dx} + X_n kf(x). \quad (8.76)$$

This equals

$$kX_0 \frac{d^n f(x)}{dx^n} + kX_1 \frac{d^{n-1} f(x)}{dx^{n-1}} + \cdots + kX_{n-1} \frac{df(x)}{dx} + kX_n f(x). \quad (8.77)$$

Since k is a factor of every term, this is

$$k\left[X_0 \frac{d^n f(x)}{dx^n} + X_1 \frac{d^{n-1} f(x)}{dx^{n-1}} + \cdots + X_{n-1} \frac{df(x)}{dx} + X_n f(x)\right]. \quad (8.78)$$

This is equal to zero since the quantity in the brackets was given equal to zero, and therefore $y = kf(x)$ is a solution since it is found on substitution to satisfy the equation.

Theorem II. If $y = f(x)$ and $y = \phi(x)$ are both solutions of a homogeneous linear differential equation, $y = f(x) + \phi(x)$ is a solution of the same equation. It will be found on substitution that $y = e^{-x}$ is a solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

Direct substitution shows that $y = e^{-2x}$ is also a solution of the same equation. If $y = e^{-x} + e^{-2x}$ is substituted, we have

$$(e^{-x} + 4e^{-2x}) + 3(-e^{-x} - 2e^{-2x}) + 2(e^{-x} + e^{-2x}) = 0$$

and the equation is satisfied. Therefore $y = e^{-x} + e^{-2x}$ is a solution. The proof of the general theorem is no more difficult than in the case of Theorem I and is left for the student.

8.12 Linear Equations with Constant Coefficients. We shall first restrict our study to the special case of linear differential equations with constant coefficients, and start by considering the homogeneous equation which can be written in the form

$$\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_ny = 0 \quad (8.79)$$

where the a's are constants. It is sometimes simpler to write this

$$D^ny + a_1D^{n-1}y + \cdots + a_{n-1}Dy + a_ny = 0 \quad (8.80)$$

where D is an operator signifying that the first derivative should be taken, D^2 the second derivative, and D^n the nth derivative. With this agreement

in mind there is no ambiguity if we write this

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = 0 \tag{8.81}$$

and, if we please, replace

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n) \text{ by } \phi(D) \tag{8.82}$$

getting $\phi(D)y = 0$. This is simply a shorthand notation for the equation above.

If we assume a solution $y = e^{mx}$ and substitute in the given equation to see if a value of m can be found to make $y = e^{mx}$ a solution, we find

$$\begin{aligned} m^n e^{mx} + a_1m^{n-1} e^{mx} + a_2m^{n-2} e^{mx} + \cdots + a_{n-1}m\, e^{mx} \\ + a_n e^{mx} = 0. \end{aligned} \tag{8.83}$$

If we multiply throughout by e^{-mx} we have the algebraic equation

$$m^n + a_1m^{n-1} + a_2m^{n-2} + \cdots + a_{n-1}m + a_n = 0 \tag{8.84}$$

which is called the characteristic equation of the differential equation. Any value of m which satisfies this equation can be used in $y = e^{mx}$ as a solution. Note that a comparison between the equation in m and $\phi(D)$ will suggest that we can indicate the equation in m by $\phi(m) = 0$. It should be borne in mind that $\phi(D)$ is an operator, and is quite different from $\phi(m)$. We look for values of m that will make $\phi(m) = 0$ whereas, if $\phi(D) = 0$, we have no differential equation to solve.

The equation $\phi(m) = 0$ is a polynomial equation of degree n, and can have n distinct roots. If $\phi(m) = 0$ has n different roots, let us designate them by $m_1, m_2, \cdots m_n$; then

$$y = e^{m_1x}, \quad y = e^{m_2x}, \quad y = e^{m_3x} \cdots y = e^{m_nx} \tag{8.85}$$

will all be solutions.

By Theorem I

$$y = C_1\, e^{m_1x}, \quad y = C_2\, e^{m_2x} \cdots y = C_n\, e^{m_nx} \tag{8.86}$$

will all be solutions.

By Theorem II

$$y = C_1\, e^{m_1x} + C_2\, e^{m_2x} + \cdots + C_n\, e^{m_nx} \tag{8.87}$$

will be a solution. This is the general solution since it contains n constants of integration $C_1, C_2 \cdots C_n$, and the given differential equation was of order n.

Example 1. $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = 0$ or, in operator form, $(D^2 + 5D + 6)y = 0$. The characteristic equation is $m^2 + 5m + 6 = 0$, which has for roots -2 and -3.

The general solution of the differential equation is $y = C_1\, e^{-2x} + C_2\, e^{-3x}$.

Example 2. $(D^3 - D^2 - 14D + 24)y = 0$. The characteristic equation is $m^3 - m^2 - 14m + 24 = 0$. This has for roots 2, 3, and -4, and the general solution is $y = C_1 e^{2x} + C_2 e^{3x} + C_3 e^{-4x}$.

Example 3. $(D^3 - D^2 + 6D)y = 0$. The characteristic equation is $m^3 - m^2 + 6m = 0$. The roots are 0, -2, and 3. The general solution is $y = C_1 e^0 + C_2 e^{-2x} + C_3 e^{3x} = C_1 + C_2 e^{-2x} + C_3 e^{3x}$.

8.13 Complex Roots. If some of the roots of the characteristic equation are complex it is usually better to take it into account in writing the general solution as follows: If m_1 and m_2 are a pair of complex roots $m_1 = a + ib$ and $m_2 = a - ib$, then instead of writing

$$C_1 e^{m_1 x} + C_2 e^{m_2 x} + \cdots \tag{8.88}$$

we write

$$C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx + \cdots. \tag{8.89}$$

That this is justified can be shown as follows:

$$K_1 e^{m_1 x} + K_2 e^{m_2 x} = K_1 e^{ax} e^{ibx} + K_2 e^{ax} e^{-ibx} \tag{8.90}$$

$$= K_1 e^{ax} (\cos bx + i \sin bx) + K_2 e^{ax} (\cos bx - i \sin bx) \tag{8.91}$$

$$= (K_1 + K_2) e^{ax} \cos bx + i(K_1 - K_2) e^{ax} \sin bx. \tag{8.92}$$

K_1 and K_2 are arbitrary constants and their sum is an arbitrary constant and we call it C_1; similarly $i(K_1 - K_2)$ is an arbitrary constant and we can designate it C_2 as we suggest above.

Example 1. $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 25y = 0$. The characteristic equation is $m^2 + 6m + 25 = 0$ and has for roots $-3 + i4$ and $-3 - i4$. Therefore the general solution is $y = C_1 e^{-3x} \cos 4x + C_2 e^{-3x} \sin 4x$.

Example 2. $\dfrac{d^3y}{dx^3} + 4\dfrac{dy}{dx} = 0$ or $(D^3 + 4D)y = 0$. The characteristic equation is $m^3 + 4m = 0$ and has for roots 0, $i2$ and $-i2$. The general solution is $y = C_1 + C_2 \cos 2x + C_3 \sin 2x$.

8.14 Multiple Roots. If some of the roots of the characteristic equation are multiple roots, there will be less than n distinct roots to the equation of the nth degree and hence the general solution as written above will contain less than n arbitrary constants. If m_1 is a double root of the characteristic equation, we find in addition to $y = C_1 e^{m_1 x}$ being a solution that $y = C_2 x e^{m_1 x}$ is also a solution of the differential equation.

If m_5 is a triple root of the characteristic equation, then

$$y = C_5 e^{m_5 x} + C_6 x e^{m_5 x} + C_7 x^2 e^{m_5 x} \tag{8.93}$$

will be a solution of the differential equation. It is apparent that a double

root supplies two arbitrary constants, a triple root, three constants. An r-fold root will supply r constants so we have n arbitrary constants for the equation of order n as we should.

Example 1. $(D^2 + 4D + 4)y = 0$. The characteristic equation is $m^2 + 4m + 4 = 0$ and has a double root -2. The general solution is therefore $y = C_1 e^{-2x} + C_2 x\, e^{-2x}$.

Example 2. $(D^3 + 4D^2 + 5D + 2)y = 0$. The characteristic equation $m^3 + 4m^2 + 5m + 2 = 0$ has a simple root -2 and a double root -1. Therefore the general solution is $y = C_1 e^{-2x} + C_2 e^{-x} + C_3 x\, e^{-x}$.

Example 3. $(D^3 + 3D^2 + 3D + 1)y = 0$. The characteristic equation $m^3 + 3m^2 + 3m + 1 = 0$ has a triple root -1. Therefore $y = C_1 e^{-x} + C_2 x\, e^{-x} + C_3 x^2 e^{-x}$ is the general solution.

Example 4. $(D^4 + 8D^2 + 16)y = 0$. The characteristic equation $m^4 + 8m^2 + 16 = 0$ has two double roots $-i2$ and $i2$. Therefore the general solution is $y = C_1 \cos 2x + C_2 \sin 2x + C_3 x \cos 2x + C_4 x \sin 2x$.

This procedure will be justified in the case of a double root, the general case of an r-fold root being left for the student to prove for himself. We *wish to show that $y = x\, e^{m_1 x}$ satisfies a given homogeneous differential* equation with constant coefficients if m_1 is a double root of the characteristic equation. Substitute $y = x\, e^{m_1 x}$ in the left-hand side of the given equation

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)y = 0 \tag{8.94}$$

and obtain

$$(D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n)(x\, e^{m_1 x}) = Z. \tag{8.95}$$

If $y = x\, e^{m_1 x}$ is a solution, Z will be zero.

It will be easier to follow if the necessary differentiation is worked out in a table as follows:

$$y = x\, e^{m_1 x} \tag{8.96}$$

$$Dy = \frac{dy}{dx} = m_1 x\, e^{m_1 x} + e^{m_1 x} \tag{8.97}$$

$$D^2 y = \frac{d^2 y}{dx^2} = m_1^2 x\, e^{m_1 x} + m_1 e^{m_1 x} + m_1 e^{m_1 x} = m_1^2 x\, e^{m_1 x} + 2m_1 e^{m_1 x} \tag{8.98}$$

$$D^3 y = m_1^3 x\, e^{m_1 x} + m_1^2 e^{m_1 x} + 2m_1^2 e^{m_1 x} = m_1^3 x\, e^{m_1 x} + 3m_1^2 e^{m_1 x} \tag{8.99}$$

$$D^n y = m_1^n x\, e^{m_1 x} + n m_1^{n-1} e^{m_1 x} \tag{8.100}$$

It is now evident that

$$Z = m_1^n x\, e^{m_1 x} + n m_1^{n-1} e^{m_1 x} + a_1 m_1^{n-1} x\, e^{m_1 x} + a_1 (n - 1) m_1^{n-2} e^{m_1 x} \\ + \cdots a_{n-1} m_1 x\, e^{m_1 x} + a_{n-1} e^{m_1 x} + a_n x\, e^{m_1 x} \tag{8.101}$$

or

$$\begin{aligned} Z = {} & (m_1^n + a_1 m_1^{n-1} + \cdots + a_{n-1}m_1 + a_n)x\, e^{m_1 x} \\ & + [n m_1^{n-1} + a_1(n-1)m_1^{n-2} + \cdots + a_{n-1}]\, e^{m_1 x}. \end{aligned} \tag{8.102}$$

A careful look at this will show that we have

$$Z = x\, e^{m_1 x}\, \phi(m_1) + e^{m_1 x}\, \phi'(m_1). \tag{8.103}$$

Now $\phi(m_1) = 0$ since m_1 is a root of $\phi(m) = 0$. Since m_1 is a double root of $\phi(m) = 0$ we know that $\phi'(m_1) = 0$. Therefore $Z = 0$, which was to be proved.

8.15 Evaluation of Integration Constants. The general solution of a differential equation of order n contains n constants of integration. In order to evaluate these n constants we require n consistent independent equations. These equations cannot be obtained from the differential equation but must be obtained directly from the physical problem. The mathematician, starting with a differential equation, considers his problem solved when he has obtained the general solution. However, the engineer, starting with a physical problem, sets up the differential equation and is required to find the particular solution which obtains for his physical problem. Therefore the engineer must evaluate the constants of integration in the general solution before he considers his problem solved.

In the case of a first order equation it is sufficient to know the value of the dependent variable corresponding to any value of the independent variable. The differential equation for the charge on a condenser C short-circuited through a resistance R is

$$R\frac{dq}{dt} + \frac{1}{C}q = 0.$$

The general solution is

$$q = K\, e^{-t/RC}.$$

If we know that the charge on the condenser at $t = T$ is equal to Q, we substitute these values in the general solution and obtain

$$Q = K\, e^{-T/RC}$$

which gives

$$K = Q\, e^{T/RC}$$

and the particular solution is

$$q = Q\, e^{(T-t)/RC}.$$

Often it is sufficient to know the value of the derivative of the dependent variable corresponding to any value of the independent variable. In the

illustration above, if we know that the current is I at time $t = T$ we must first differentiate the general solution to obtain an expression for the current

$$i = \frac{dq}{dt} = -\frac{K}{RC}\, e^{-t/RC}.$$

Substitution in the equation for current gives us

$$I = -\frac{K}{RC}\, e^{-T/RC}.$$

The integration constant K is found by solving the above equation

$$K = -IRC\, e^{T/RC}$$

and the particular solution is

$$q = -IRC\, e^{(T-t)/RC}.$$

In the case of a second order equation where there are two constants of integration, we require two equations for their determination. It is sufficient to know the value of the dependent variable corresponding to two values of the independent variable, or to know the value of the dependent variable and the value of one of its derivatives at a common value or at different values of the independent variable. It is clear that there are many ways of finding sufficient equations. The extension to three or more constants is obvious.

In general there must be as many conditions known as there are constants to be evaluated. These conditions must be independent and consistent. The examples that follow illustrate what happens if the conditions used are dependent (example 1), inconsistent (example 2), and independent and consistent (examples 3 and 4).

Example 1. The second order equation $D^2y = 0$ has for its general solution

$$y = C_1x + C_2.$$

If we use the relations $y = 5$ at $x = 1$, and $D^3y = 0$ at $x = 3$, the problem cannot be solved because these relations are not independent. We know from the given differential equation that $D^3y = 0$ for all values of x and therefore the statement that $D^3y = 0$ for $x = 3$ tells us nothing new.

Example 2. The equation $D^2y - 4y = 0$ has for its general solution

$$y = C_1\, e^{2x} + C_2\, e^{-2x}.$$

If we try to solve for C_1 and C_2 to satisfy the conditions $y = 1$ at $x = 1$, and $D^2y = 1$ at $x = 1$, we find no solution is possible because we have from the differential equation that $D^2y = 4$ when $y = 1$ and therefore cannot be equal to 1 as required by the stated conditions.

Example 3. The general solution of the equation $(D^2 + 3D + 2)y = 0$ is

$$y = C_1 e^{-x} + C_2 e^{-2x}.$$

If $y = 5$ at $x = 0$, and $y = 2$ at $x = 1$, we have

$$\begin{aligned} C_1 + C_2 &= 5, \\ C_1 e^{-1} + C_2 e^{-2} &= 2. \end{aligned}$$

Solving these for C_1 and C_2 using determinant notation we have

$$C_1 = \frac{\begin{vmatrix} 5 & 1 \\ 2 & e^{-2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ e^{-1} & e^{-2} \end{vmatrix}}, \qquad C_2 = \frac{\begin{vmatrix} 1 & 5 \\ e^{-1} & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ e^{-1} & e^{-2} \end{vmatrix}}.$$

Example 4. If in the preceding example we have $Dy = 3$ at $x = 0$, and $D^3y = 8$ at $x = 2$, we must differentiate first to obtain the first and third derivatives.

$$\begin{aligned} Dy &= -C_1 e^{-x} - 2C_2 e^{-2x}, \\ D^2y &= C_1 e^{-x} + 4C_2 e^{-2x}, \\ D^3y &= -C_1 e^{-x} - 8C_2 e^{-2x}. \end{aligned}$$

Substituting the above conditions we have

$$\begin{aligned} -C_1 e^{-2} - 8C_2 e^{-4} &= 8, \\ -C_1 - 2C_2 &= 3. \end{aligned}$$

These equations give for C_1 and C_2

$$C_1 = \frac{\begin{vmatrix} 8 & -8e^{-4} \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} -e^{-2} & -8e^{-4} \\ -1 & -2 \end{vmatrix}}. \qquad C_2 = \frac{\begin{vmatrix} -e^{-2} & 8 \\ -1 & 3 \end{vmatrix}}{\begin{vmatrix} -e^{-2} & -8e^{-4} \\ -1 & -2 \end{vmatrix}}.$$

8.16 Integration by Means of Series. If the dependent variable y, is such a function of x that it can be expressed as a power series, it is sometimes convenient to use the following method: Write

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots. \tag{8.104}$$

Substitute this in the given equation and obtain an identity in x. If this gives equations that can be solved for A_0, A_1, etc., we have a solution to the problem in terms of a power series, or MacLaurin's series. This method is particularly useful for linear equations where the coefficients are no worse than polynomials in x.

Consider the following example: Given the equation $(D - 1)y = 0$. Substitute

$$y = A_0 + A_1x + A_2x^2 + \cdots \tag{8.105}$$

$$Dy = A_1 + 2A_2x + 3A_3x^2 + \cdots \tag{8.106}$$

in the given equation. This gives

$$(D-1)y = (A_1 - A_0) + (2A_2 - A_1)x + (3A_3 - A_2)x^2 + \cdots + (nA_n - A_{n-1})x^{n-1} + \cdots = 0. \tag{8.107}$$

This is zero for every value of x; therefore we equate the coefficient of each power of x to zero

$$\begin{aligned} A_1 - A_0 &= 0. \\ 2A_2 - A_1 &= 0. \\ nA_n - A_{n-1} &= 0. \end{aligned} \tag{8.108}$$

These equations can be solved, giving

$$\begin{aligned} A_1 &= A_0. \\ A_2 &= \frac{A_1}{2} = \frac{A_0}{2!}. \\ A_3 &= \frac{A_2}{3} = \frac{A_0}{3!}. \\ A_n &= \frac{A_{n-1}}{n} = \frac{A_0}{n!}. \end{aligned} \tag{8.109}$$

Therefore we have for a solution

$$y = A_0\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right]. \tag{8.110}$$

The series in the brackets in equation (8.110) we recognize as e^x, and our solution is $y = A_0 e^x$, where A_0 is the integration constant.

As a second example consider the equation

$$x\frac{dy}{dx} - y = 0.$$

Write as before

$$\begin{aligned} y &= A_0 + A_1x + A_2x^2 + \cdots \\ \frac{dy}{dx} &= A_1 + 2A_2x + 3A_3x^2 + \cdots \\ x\frac{dy}{dx} &= A_1x + 2A_2x^2 + 3A_3x^3 + \cdots \end{aligned} \tag{8.111}$$

We now substitute in the original equation and obtain

$$x\frac{dy}{dx} - y = -A_0 - (A_1 - A_1)x - (A_2 - 2A_2)x^2 - (A_3 - 3A_3)x^3 \cdots - (A_n - nA_n)x^n \cdots = 0. \tag{8.112}$$

This is true for every value of x; therefore we may equate the coefficient

of each power of x to zero and we have

$$\begin{aligned} A_0 &= 0. \\ A_2 &= 0. \\ A_3 &= 0. \\ A_n &= 0, \quad (n \neq 1). \end{aligned} \tag{8.113}$$

This leaves us with

$$y = A_1 x$$

which is the general solution and A_1 is the constant of integration.

As a third example, consider the equation

$$x^2 \frac{dy}{dx} - y = 0.$$

Write as before

$$\begin{aligned} y &= A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots \\ x^2 \frac{dy}{dx} &= A_1 x^2 + 2A_2 x^3 + 3A_3 x^4 + \cdots \end{aligned} \tag{8.114}$$

Substituting in the original equation we have

$$\begin{aligned} x^2 \frac{dy}{dx} - y = -A_0 - A_1 x &+ (A_1 - A_2)x^2 \\ &+ (2A_2 - A_3)x^3 + \cdots = 0. \end{aligned} \tag{8.115}$$

This is an identity in x; therefore we equate the coefficient of each power of x to zero and find

$$\begin{aligned} A_0 &= 0. \\ A_1 &= 0. \\ A_2 &= 0. \\ A_n &= 0. \end{aligned} \tag{8.116}$$

This gives for a solution $y = 0$. Now $y = 0$ does satisfy the given equation. It is, in fact, a particular solution but not the general solution.

To find what the trouble is in this case let us solve the given equation by separating the variables; we have

$$\frac{dy}{y} = \frac{dx}{x^2}. \tag{8.117}$$

This can be integrated now, giving

$$\ln \frac{y}{C} = -\frac{1}{x} \tag{8.118}$$

which can be written

$$\frac{y}{C} = e^{-1/x}.$$

The general solution can now be written

$$y = C\,e^{-1/x}. \qquad (8.119)$$

The difficulty is that the quantity

$$e^{-1/x}$$

cannot be expressed as a power series; therefore

$$C\,e^{-1/x}$$

cannot be expressed by a power series except for the special case where $C = 0$.

It sometimes happens that the first few coefficients in the series for y

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots \qquad (8.104)$$

are equal to zero. In this case it is simpler to write

$$y = x^r(A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots) \qquad (8.120)$$

where r, A_0, A_1, etc., are to be determined. In this case it may happen that r is not an integer. Therefore the use of equation (8.120) will sometimes result in a solution where equation (8.104) will not. This method [equation (8.120)] is used in Chapter 10 to find a solution of Bessel's equation (see section 10.10).

PROBLEMS ON CHAPTER 8

1. Find the differential equation of the family of circles having centers at the origin.

2. Find the differential equation of the family of unit circles having centers on the x axis.

3. Find the differential equation of the family of unit circles having centers on the y axis.

4. Find the differential equation of the family of all straight lines.

5. Find the differential equation of the family of all circles.

Find the general solution of each of the following equations:

6. $y^2\,dx + xy^2\,dx - x^3\,dy = 0$

7. $x^2\,dx + 3y\,dy = 0$

8. $\sec x \cos^2 y\,dx - 3\cos x \sin y\,dy = 0$

9. $x\,dy - y\,dx + 4x^2\,dx = 0$

10. $x\,dy - y\,dx + 4y^2\,dy = 0$

11. $x\,dy + y\,dx + 3xy\,dy = 0$

12. $\dfrac{dy}{dx} + \dfrac{2y}{x+1} = (x+1)^4$

13. $(x + x^4)\dfrac{dy}{dx} + 3x^3y = 6$

14. $x\dfrac{dy}{dx} + y = x^2$

15. $dx + x^2\,dy - xy\,dx = 0$

16. $(y^2 - xy)\,dx + x^2\,dy = 0$

17. $(x + y)\,dx - (x - y)\,dy = 0$

18. $\dfrac{d^2y}{dx^2} + 3x = 0$

19. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$

20 $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2x^2\dfrac{dy}{dx} = 0$

21. $2y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$

22. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 + \dfrac{dy}{dx} = 0$

23. Show by substitution that $y = e^x$ is a solution of the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 0.$$

24. Show that $y = e^{4x}$ is a solution of the equation in problem 23.

25. Show that $y = 4e^{4x}$ is a solution of the equation in problem 23.

26. Show that $y = A\,e^{4x}$ is a solution of the equation in problem 23 for any value of A whatsoever.

27. Show that $y = e^x + e^{4x}$ is a solution of the equation in problem 23.

28. Show that $y = K\,e^x + M\,e^{4x}$ is a solution of the equation in problem 23 for any value of K and for any value of M.

29. Show that $y = x$ is a solution of the equation

$$\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} = 0.$$

30. Show that $y = Kx + B$ is a solution of the equation in problem 29.

31. Show that $y = x\,e^{-2x}$ is a solution of the equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

32. Show that $y = Ax\,e^{-2x} + B\,e^{-2x}$ is a solution of the equation in problem 31 for any values of A and B.

Find the general solution of each of the following equations:

33. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

34. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

35. $\frac{d^4y}{dx^4} + 4\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} = 0$

36. $\frac{d^3y}{dx^3} + y = 0$

37. $\frac{d^2y}{dx^2} + y = 0$

38. $\frac{d^2y}{dx^2} - y = 0$

39. Find the particular solution to problem 33 passing through the points $x = 0, y = 1; x = 1, y = 0$.

40. Find the particular solution to problem 33 passing through the points $x = 0, y = 1; x = 1, y = 1$.

41. Find the particular solution to problem 37 passing through the points $x = 0, y = 1; x = 1, y = 2$.

42. Find the particular solution to problem 38 passing through the points $x = 0, y = 1; x = 1, y = 2$.

CHAPTER 9

DIFFERENTIAL EQUATIONS (*continued*)

A. Complete Linear Differential Equations with Constant Coefficients

9.1 Introduction. Consider an equation of the form

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = X \tag{9.1}$$

where X is a function of the independent variable x. To find the general solution we have two conditions to fulfill. If y is the general solution, it not only satisfies the equation for any value of x but it also contains n independent constants of integration, since the equation is of order n. The following theorem enables us to consider these two conditions separately for this type of equation.

Theorem: If y_2 is a solution of the complete linear differential equation (9.1), and y_1 is a solution of the homogeneous linear differential equation obtained by replacing X on the right of equation (9.1) by zero, then $y = y_1 + y_2$ is a solution of the complete linear differential equation (9.1).

This theorem can be proved as follows: Substitute $y = y_1 + y_2$ in the left side of (9.1). We have

$$\begin{aligned}(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)(y_1 + y_2)\\ = (D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y_1\\ + (D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y_2.\end{aligned} \tag{9.2}$$

The first term on the right side of equation (9.2) is zero and the second term is equal to X by hypothesis. Therefore, $y_1 + y_2$ satisfies the given equation (9.1).

The solution of the complete linear differential equation is now replaced by the solution of two equations.

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y_1 = 0. \tag{9.3}$$

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y_2 = X. \tag{9.4}$$

We saw in the preceding chapter how to solve equation (9.3) and obtain y_1 as a function of x containing n constants of integration. Since $y = y_1 + y_2$ must contain n constants of integration, and since y_1 contains that many, y_2 need not contain any. When we solve equation (9.4) for y_2, we do not

look for integration constants. y_1, the solution of equation (9.3), is called the **complementary function** and y_2, the solution of equation (9.4), is called a **particular solution,** or **particular integral.**

There are several standard methods of obtaining the particular solution, such as repeated integration, partial fractions, undetermined coefficients, variation of parameters, etc. We shall not try to cover all possible methods. Therefore, for a more complete treatment, we refer the interested student to texts devoted to differential equations.

9.2 Method of Repeated Integration. This method will always work formally. One may be prevented from using it, however, because of complicated integrals.

Let $m_1, m_2, m_3, \cdots m_n$ be the roots of the characteristic equation (9.5) of equation (9.3)

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1}m + a_n = 0. \tag{9.5}$$

Then the differential equation (9.4) can be written

$$(D - m_1)(D - m_2)(D - m_3) \cdots (D - m_n)y_2 = X. \tag{9.6}$$

Let

$$(D - m_2)(D - m_3) \cdots (D - m_n)y_2 = z. \tag{9.7}$$

Then equation (9.6) becomes

$$(D - m_1)z = X. \tag{9.8}$$

This is a first order equation and can be integrated using an integrating factor. Multiply equation (9.8) by e^{-m_1x}.

$$\left(\frac{dz}{dx} - m_1 z\right) e^{-m_1x} = X\, e^{-m_1x}. \tag{9.9}$$

Since

$$\frac{d}{dx}(z\, e^{-m_1x}) = \frac{dz}{dx} e^{-m_1x} - m_1 z\, e^{-m_1x}$$

we have

$$z\, e^{-m_1x} = \int X\, e^{-m_1x}\, dx \tag{9.10}$$

or, finally,

$$z = e^{m_1x} \int X\, e^{-m_1x}\, dx. \tag{9.11}$$

We have omitted the constant of integration in (9.10) and (9.11) since it is taken care of in the complementary function. Equations (9.7) and

(9.11) give us

$$(D - m_2)(D - m_3) \cdots (D - m_n)y_2 = e^{m_1 x} \int X\, e^{-m_1 x}\, dx. \tag{9.12}$$

We treat this as we did equation (9.6).

Let

$$(D - m_3) \cdots (D - m_n)y_2 = w. \tag{9.13}$$

We have then

$$(D - m_2)w = e^{m_1 x} \int X\, e^{-m_1 x}\, dx. \tag{9.14}$$

Comparing this with (9.8), we see we can write the solution

$$w = e^{m_2 x} \int e^{-m_2 x}\, e^{m_1 x} \int X\, e^{-m_1 x}\, dx\, dx. \tag{9.15}$$

The next step would give

$$e^{m_3 x} \int e^{-m_3 x}\, e^{m_2 x} \int e^{-m_2 x}\, e^{m_1 x} \int X\, e^{-m_1 x}\, dx\, dx\, dx. \tag{9.16}$$

We have gone far enough to indicate the general rule. It is also evident that the integration problem becomes more involved as the order of the equation is higher.

Example. $(D^2 + 7D + 12)y = 2e^{2x}$ The complementary function is $y_1 = C_1\, e^{-3x} + C_2\, e^{-4x}$. The formula is

$$\begin{aligned} y_2 &= e^{-4x} \int e^{4x}\, e^{-3x} \int 2e^{2x}\, e^{3x}\, dx\, dx \\ &= e^{-4x} \int e^{x} \int 2e^{5x}\, dx\, dx \\ &= e^{-4x} \int e^{x}\, \frac{2}{5} e^{5x}\, dx = \frac{2e^{-4x}}{5} \int e^{6x}\, dx \\ &= \frac{2e^{-4x}}{5}\, \frac{e^{6x}}{6} = \frac{e^{2x}}{15}. \end{aligned}$$

The general solution is

$$y = C_1\, e^{-3x} + C_2\, e^{-4x} + \tfrac{1}{15}\, e^{2x}.$$

This example is worked below by the following method.

9.3 Method of Undetermined Coefficients. Although this method cannot always be used, it is usually the simplest. It can be employed when X on the right of the equation and all its derivatives include only

a finite number of forms. For example, suppose $X = x^3$. The first derivative is $3x^2$, the second $6x$, the third is 6, the fourth and all higher derivatives are zero. In this case there are only four forms: x^3, x^2, x, constant. If $X = \sin x$, the first derivative is $\cos x$, the second is $-\sin x$. There are only two forms: $\sin x$ and $\cos x$. If we have

$$X = \frac{1}{x},$$

$$\frac{dX}{dx} = -\frac{1}{x^2}, \qquad \frac{d^2X}{dx^2} = \frac{2}{x^3}, \ldots$$

Each derivative is a new form and there is an infinite number of forms in this case. Hence the present method cannot be used. It will be found that this method can be used where X is made up of sums and products of terms such as

$$x^h, \qquad e^{px}, \qquad \sin qx, \qquad \cos ux,$$

where h is zero or a positive integer and p, q, and u are any constants.

The method consists of taking all the forms in X and all the forms that can be found by differentiation, multiplying each form by a coefficient to be determined, and substituting the sum for y_2 in the equation. The result is an identity in x; therefore, the coefficients of corresponding terms on both sides of the equal sign can be equated. This procedure will give a set of simultaneous algebraic equations from which the undetermined coefficients can be evaluated. We shall illustrate with several examples.

Example 1. $(D^2 + 7D + 12)y = 2e^{2x}$. The complementary function is found to be $y_1 = C_1 e^{-3x} + C_2 e^{-4x}$.

Let

$$\begin{aligned} y_2 &= A\, e^{2x} \\ Dy_2 &= 2A\, e^{2x} \\ 7Dy_2 &= 14A\, e^{2x} \\ D^2y_2 &= 4A\, e^{2x} \\ (4A + 14A + 12A)\, e^{2x} &= 2e^{2x} \\ 30A &= 2 \\ A &= \tfrac{1}{15} \end{aligned}$$

The general solution is $y = C_1 e^{-3x} + C_2 e^{-4x} + \frac{1}{15} e^{2x}$.

Example 2. $(D^2 + D - 12)y = x\, e^x$. The complementary function is $y_1 = C_1 e^{3x} + C_2 e^{-4x}$.

Let

$$\begin{aligned} y_2 &= A_1 x\, e^x + A_2\, e^x \\ Dy_2 &= A_1 x\, e^x + A_1\, e^x + A_2\, e^x \\ D^2y_2 &= A_1 x\, e^x + 2A_1\, e^x + A_2\, e^x \end{aligned}$$

$$(A_1 x\, e^x + 2A_1 e^x + A_2 e^x) + (A_1 x\, e^x + A_1 e^x + A_2 e^x) - 12(A_1 x\, e^x + A_2 e^x) = x\, e^x$$

$$(A_1 + A_1 - 12A_1)x\, e^x + (2A_1 + A_2 + A_1 + A_2 - 12A_2)e^x = x\, e^x$$

Therefore,

$$-10A_1 = 1 \qquad A_1 = -\frac{1}{10}$$

$$3A_1 - 10A_2 = 0 \qquad A_2 = \frac{3A_1}{10} = -\frac{3}{100}$$

The general solution is

$$y = C_1 e^{3x} + C_2 e^{-4x} - 0.1x e^x - 0.03e^x.$$

Example 3. $(D^2 + 5D + 6)y = x^2 + x$. The complementary function is $y_1 = C_1 e^{-2x} + C_2 e^{-3x}$.

Let

$$\begin{aligned} y_2 &= A_1x^2 + A_2x + A_3 \\ Dy_2 &= 2A_1x + A_2 \\ D^2y_2 &= 2A_1 \end{aligned}$$

$$2A_1 + 5(2A_1x + A_2) + 6(A_1x^2 + A_2x + A_3) = x^2 + x$$

$$6A_1x^2 + (10A_1 + 6A_2)x + (2A_1 + 5A_2 + 6A_3) = x^2 + x$$

Therefore,

$$6A_1 = 1 \qquad A_1 = \frac{1}{6}$$

$$10A_1 + 6A_2 = 1 \qquad A_2 = \frac{1}{6} - \frac{10A_1}{6} = -\frac{1}{9}$$

$$2A_1 + 5A_2 + 6A_3 = 0 \qquad A_3 = \frac{1}{27}$$

The general solution is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{x^2}{6} - \frac{x}{9} + \frac{1}{27}$$

9.4 Superposition Theorem. The following theorem is valuable in that its use tends to reduce the chances of making numerical mistakes while evaluating the coefficients.

Theorem: If y_2 satisfies the differential equation

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y_2 = X_2 \tag{9.17}$$

and if y_3 satisfies the equation

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y_3 = X_3 \tag{9.18}$$

then $y = y_2 + y_3$ will satisfy the equation

$$(D^n + a_1D^{n-1} + \cdots + a_{n-1}D + a_n)y = X_2 + X_3. \tag{9.19}$$

This theorem can be proved by substituting $y = y_2 + y_3$ in equation (9.19) and then using equations (9.17) and (9.18) to show that (9.19) is satisfied.

Example 4. $(D^2 + 5D + 6)y = e^{2x} + x^2 + x$. We have already done part of the work in example 3, section 9.3. All we need do now is to let $y_3 = A_4 e^{2x}$.

$$4A_4 e^{2x} + 10A_4 e^{2x} + 6A_4 e^{2x} = e^{2x}$$
$$20A_4 = 1$$
$$A_4 = \frac{1}{20}$$

Therefore the general solution is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{x^2}{6} - \frac{x}{9} + \frac{1}{27} + \frac{1}{20} e^{2x}.$$

Note that the superposition principle is not used in cases like example 3, where $X = x^2 + x$, nor would it be used for $X = \sin x + 3 \cos x$ nor $X = x^2 \sin x + \cos x$. It would be used, however, for $X = e^x \sin x + \cos x$.

9.5 Exceptions. There are some cases where the method already discussed must be varied. Suppose we use the above method to solve the equation $(D^2 + 3D + 2)y = e^{-2x}$. Not knowing that this is a special case we let $y_2 = A\, e^{-2x}$ and substitute $Dy_2 = -2A\, e^{-2x}$, $D^2y_2 = 4A\, e^{-2x}$. Therefore, we have $(4A - 6A + 2A)\, e^{-2x} = e^{-2x}$ or $0 = 1$. If we had obtained the complementary function first, $y_1 = C_1 e^{-x} + C_2 e^{-2x}$, as is done in each example above, we would see that the term on the right of the equation is the same as one term in the complementary function and would know that we have a special case. The way out of the difficulty is to let $y_2 = Ax\, e^{-2x}$; then $Dy_2 = -2Ax\, e^{-2x} + A\, e^{-2x}$, $D^2y_2 = 4Ax\, e^{-2x} - 4A\, e^{-2x}$. Therefore, when we substitute, we have

$$(4Ax\, e^{-2x} - 4A\, e^{-2x}) + 3(-2Ax\, e^{-2x} + A\, e^{-2x}) + 2Ax\, e^{-2x} = e^{-2x},$$
$$(4A - 6A + 2A)x\, e^{-2x} + (-4A + 3A)\, e^{-2x} = e^{-2x}.$$

Therefore, $-A = 1$ or $A = -1$, and the general solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} - x\, e^{-2x}.$$

The rule to follow for the special cases is really very simple after a few practice problems have been worked. If some term in X on the right of equation (9.1) is of the form $x^h e^{px}$, where h is zero or a positive integer and p is a root of the characteristic equation, then include $x^{h+1} e^{px}$ in y_2 and omit e^{px} if p is a simple root. Include $x^{h+2} e^{px}$, and of course $x^{h+1} e^{px}$, and omit $x\, e^{px}$ and e^{px} in y_2 if p is a double root, etc. The first and third examples which follow are interesting in that $p = 0$. The same rule applies when p is complex; see the fourth example.

Example 1. $(D^2 + 2D)y = 4$. The complementary function is $y_1 = C_1 + C_2 e^{-2x}$. Let $y_2 = Ax$; then $Dy_2 = A$, $D^2y_2 = 0$, and we have

$$0 + 2A = 4 \quad \text{or} \quad A = 2.$$

The general solution is $y = C_1 + C_2 e^{-2x} + 2x$.

Example 2. $(D^2 + 4D + 4)y = e^{-2x}$. The complementary function is $y_1 = C_1 e^{-2x} + C_2 x e^{-2x}$. Let $y_2 = Ax^2 e^{-2x}$, $Dy_2 = -2Ax^2 e^{-2x} + 2Ax e^{-2x}$, $D^2 y_2 = 4Ax^2 e^{-2x} - 8Ax e^{-2x} + 2A e^{-2x}$. Substituting, we have $(4A - 8A + 4A)x^2 e^{-2x} - (8A - 8A)x e^{-2x} + 2A e^{-2x} = e^{-2x}$. Therefore, $A = 0.5$ and the general solution is $y = C_1 e^{-2x} + C_2 x e^{-2x} + 0.5 x^2 e^{-2x}$.

Example 3. $(D^4 + 4D^3)y = x^2$. The complementary function is $y_1 = C_1 e^{-4x} + C_2 + C_3 x + C_4 x^2$. Let $y_2 = A_1 x^5 + A_2 x^4 + A_3 x^3$.

$$\begin{aligned} D^3 y_2 &= 60A_1 x^2 + 24A_2 x + 6A_3. \\ D^4 y_2 &= 120A_1 x + 24A_2. \end{aligned}$$

Substituting, we have

$$240A_1 x^2 + (120A_1 + 96A_2)x + (24A_2 + 24A_3) = x^2.$$

Therefore,

$$\begin{aligned} 240A_1 &= 1 & \qquad A_1 &= \frac{1}{240} \\ 120A_1 + 96A_2 &= 0 & \qquad A_2 &= \frac{-1}{192} \\ 24A_2 + 24A_3 &= 0 & \qquad A_3 &= \frac{1}{192} \end{aligned}$$

The general solution is

$$y = C_1 e^{-4x} + C_2 + C_3 x + C_4 x^2 + \frac{x^3}{192} - \frac{x^4}{192} + \frac{x^5}{240}.$$

Example 4. $(D^2 + 4)y = \sin 2x$. The complementary function is

$$y_1 = C_1 \sin 2x + C_2 \cos 2x.$$

Let

$$\begin{aligned} y_2 &= A_1 x \sin 2x + A_2 x \cos 2x \\ Dy_2 &= 2A_1 x \cos 2x + A_1 \sin 2x - 2A_2 x \sin 2x + A_2 \cos 2x \\ D^2 y_2 &= -4A_1 x \sin 2x + 4A_1 \cos 2x - 4A_2 x \cos 2x - 4A_2 \sin 2x \\ (D^2 + 4)y &= -4A_1 x \sin 2x + 4A_1 \cos 2x - 4A_2 x \cos 2x - 4A_2 \sin 2x \\ &\quad + 4A_1 x \sin 2x + 4A_2 x \cos 2x = \sin 2x \end{aligned}$$

or

$$4A_1 \cos 2x - 4A_2 \sin 2x = \sin 2x.$$

Therefore,

$$A_1 = 0, \quad A_2 = -0.25.$$

The general solution is $y = C_1 \sin 2x + C_2 \cos 2x - 0.25x \cos 2x$.

B. Simultaneous Linear Differential Equations with Constant Coefficients.

9.6 Introduction. In problems where there are several dependent variables and one independent variable we may be able to relate the several

variables to one another by means of linear differential equations. Indeed, this is commonly the case in electric circuit problems where the currents in the various branches are dependent variables and time is the independent variable. The procedure suggested is a generalization of the method for one equation. Since the procedure is rather long it is better to work a problem and describe the necessary steps as the problem is solved rather than to list the rules and then apply them.

9.7 Method for Simultaneous Homogeneous Equations. Consider the following set of simultaneous equations:

$$\frac{dx}{dt} + 2x - 2z = 0, \tag{9.20}$$

$$\frac{dy}{dt} + 5y - 2z = 0, \tag{9.21}$$

$$x + y + z = 0. \tag{9.22}$$

Make the following substitution: $x = X_1 e^{mt}$, $y = Y_1 e^{mt}$, $z = Z_1 e^{mt}$. We have

$$mX_1 e^{mt} + 2X_1 e^{mt} - 2Z_1 e^{mt} = 0, \tag{9.23}$$

$$mY_1 e^{mt} + 5Y_1 e^{mt} - 2Z_1 e^{mt} = 0, \tag{9.24}$$

$$X_1 e^{mt} + Y_1 e^{mt} + Z_1 e^{mt} = 0. \tag{9.25}$$

These equations can be divided through by e^{mt}.

$$(m + 2)X_1 - 2Z_1 = 0, \tag{9.26}$$

$$(m + 5)Y_1 - 2Z_1 = 0, \tag{9.27}$$

$$X_1 + Y_1 + Z_1 = 0. \tag{9.28}$$

These are three homogeneous algebraic equations in three unknowns. Unless the determinant of the coefficients is zero, X_1, Y_1, and Z_1 must all be zero. Let us see if we can make the determinant zero.

$$\begin{vmatrix} (m + 2) & 0 & -2 \\ 0 & (m + 5) & -2 \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{9.29}$$

When we expand the determinant symbol, equation (9.29) becomes

$$m^2 + 11m + 24 = 0. \tag{9.30}$$

This equation is the characteristic equation of the set of simultaneous equations. Since it is of the second degree our set of differential equations is of the second order, and we expect two arbitrary constants in the general solution.

Since the roots of equation (9.30) are -3 and -8, expressions of the form $x = X_1 e^{-3t}$, $y = Y_1 e^{-3t}$; $z = Z_1 e^{-3t}$ and $x = X_2 e^{-8t}$, $y = Y_2 e^{-8t}$, $z = Z_2 e^{-8t}$ should satisfy equations (9.20) to (9.22). If we substitute $x = X_1 e^{-3t}$, $y = Y_1 e^{-3t}$, $z = Z_1 e^{-3t}$ in equations (9.20) to (9.22), we have

$$-3X_1 e^{-3t} + 2X_1 e^{-3t} - 2Z_1 e^{-3t} = 0, \tag{9.31}$$

$$-3Y_1 e^{-3t} + 5Y_1 e^{-3t} - 2Z_1 e^{-3t} = 0, \tag{9.32}$$

$$X_1 e^{-3t} + Y_1 e^{-3t} + Z_1 e^{-3t} = 0. \tag{9.33}$$

If we now divide through by e^{-3t}, we have

$$-X_1 - 2Z_1 = 0, \tag{9.34}$$

$$2Y_1 - 2Z_1 = 0, \tag{9.35}$$

$$X_1 + Y_1 + Z_1 = 0. \tag{9.36}$$

The determinant of the coefficients

$$\begin{vmatrix} -1 & 0 & -2 \\ 0 & 2 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

is of rank 2. We can solve for X_1 and Y_1 in terms of Z_1 and obtain

$$X_1 = -2Z_1, \tag{9.37}$$

$$Y_1 = Z_1. \tag{9.38}$$

Therefore, a solution of the given equations is

$$x = -2Z_1 e^{-3t} \tag{9.39}$$

$$y = Z_1 e^{-3t}, \tag{9.40}$$

$$z = Z_1 e^{-3t}. \tag{9.41}$$

Similarly, if we substitute $x = X_2 e^{-8t}$, $y = Y_2 e^{-8t}$, $z = Z_2 e^{-8t}$ in equations (9.20) to (9.22), we have

$$-8X_2 e^{-8t} + 2X_2 e^{-8t} - 2Z_2 e^{-8t} = 0, \tag{9.42}$$

$$-8Y_2 e^{-8t} + 5Y_2 e^{-8t} - 2Z_2 e^{-8t} = 0, \tag{9.43}$$

$$X_2 e^{-8t} + Y_2 e^{-8t} + Z_2 e^{-8t} = 0. \tag{9.44}$$

These can be divided by e^{-8t}, giving

$$-6X_2 - 2Z_2 = 0, \tag{9.45}$$

$$-3Y_2 - 2Z_2 = 0, \tag{9.46}$$

$$X_2 + Y_2 + Z_2 = 0. \tag{9.47}$$

The determinant of the coefficients

$$\begin{vmatrix} -6 & 0 & -2 \\ 0 & -3 & -2 \\ 1 & 1 & 1 \end{vmatrix}$$

is of rank 2. We can solve for X_2 and Y_2 in terms of Z_2.

$$X_2 = -\tfrac{1}{3}Z_2, \tag{9.48}$$

$$Y_2 = -\tfrac{2}{3}Z_2. \tag{9.49}$$

Therefore, a solution of the given equations is

$$x = -\tfrac{1}{3}Z_2\, e^{-8t}, \tag{9.50}$$

$$y = -\tfrac{2}{3}Z_2\, e^{-8t}, \tag{9.51}$$

$$z = Z_2\, e^{-8t}. \tag{9.52}$$

Since the given equations are linear, a sum of two solutions is a solution. We have for the general solution

$$x = -2Z_1\, e^{-3t} - \tfrac{1}{3}Z_2\, e^{-8t}, \tag{9.53}$$

$$y = Z_1\, e^{-3t} - \tfrac{2}{3}Z_2\, e^{-8t}, \tag{9.54}$$

$$z = Z_1\, e^{-3t} + Z_2\, e^{-8t}. \tag{9.55}$$

The constants Z_1 and Z_2 are the two integration constants which we expect to find in the general solution of a second order set of differential equations.

9.8 Multiple Roots of the Characteristic Equation. We saw that, in the case of a single equation, multiple roots of the characteristic equation called for special consideration. We find that multiple roots of the characteristic equation of a set of simultaneous differential equations also require special study.

The rules to follow are (1) If, when we substitute the multiple root into the equations, the determinant of the coefficients is of rank one less than the order of the determinant, we treat the problem just as we did for one equation, i.e., use e^{mt} and $t\,e^{mt}$; (2) If the rank of the determinant is 2 less than the order of the determinant when we substitute a double root, we use only e^{mt} and the double root will be taken care of automatically. It is apparent that there are many possible situations that may arise. We present examples of the two possible cases of a double root.

Example 1.

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 2x - 6y = 0. \tag{9.56}$$

$$6y - 3z = 0. \tag{9.57}$$

$$x + y + z = 0. \tag{9.58}$$

These equations can be written in operator form if preferred.

$$(D^2 + 4D + 2)x - 6y = 0, \tag{9.59}$$

$$6y - 3z = 0. \tag{9.60}$$

$$x + y + z = 0. \tag{9.61}$$

First substitute $x = X_1 e^{mt}$, $y = Y_1 e^{mt}$, $z = Z_1 e^{mt}$, and then divide by e^{mt}, giving

$$(m^2 + 4m + 2)X_1 - 6Y_1 = 0, \tag{9.62}$$

$$6Y_1 - 3Z_1 = 0, \tag{9.63}$$

$$X_1 + Y_1 + Z_1 = 0. \tag{9.64}$$

The characteristic equation is obtained from the coefficients of equations (9.62) to (9.64) by equating the determinant of the coefficients to zero.

$$\begin{vmatrix} (m^2 + 4m + 2) & -6 & 0 \\ 0 & 6 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{9.65}$$

On expansion of the determinant symbol, this equation is

$$m^2 + 4m + 4 = 0. \tag{9.66}$$

Again we have a second order system of equations. Equation (9.66) has a double root, -2.

Substitute $x = X_1 e^{-2t}$, $y = Y_1 e^{-2t}$, $z = Z_1 e^{-2t}$ in equations (9.56) to (9.58) and obtain

$$4X_1 e^{-2t} - 8X_1 e^{-2t} + 2X_1 e^{-2t} - 6Y_1 e^{-2t} = 0, \tag{9.67}$$

$$6Y_1 e^{-2t} - 3Z_1 e^{-2t} = 0, \tag{9.68}$$

$$X_1 e^{-2t} + Y_1 e^{-2t} + Z_1 e^{-2t} = 0. \tag{9.69}$$

These equations, when divided by e^{-2t}, become

$$-2X_1 - 6Y_1 = 0, \tag{9.70}$$

$$6Y_1 - 3Z_1 = 0, \tag{9.71}$$

$$X_1 + Y_1 + Z_1 = 0. \tag{9.72}$$

The determinant of the coefficients

$$\begin{vmatrix} -2 & -6 & 0 \\ 0 & 6 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

is of rank 2. We can solve for X_1 and Z_1 in terms of Y_1.

$$X_1 = -3Y_1. \tag{9.73}$$

$$Z_1 = 2Y_1. \tag{9.74}$$

A solution of the given equations is

$$x = -3Y_1 e^{-2t}, \tag{9.75}$$

$$y = Y_1 e^{-2t}, \tag{9.76}$$

$$z = 2Y_1 e^{-2t}. \tag{9.77}$$

Since our third order determinant was of rank 2, we must next substitute

$$x = X_2 t\, e^{-2t},$$
$$y = Y_2 t\, e^{-2t},$$
$$z = Z_2 t\, e^{-2t},$$

in the original equations; we require

$$Dx = -2X_2 t\, e^{-2t} + X_2\, e^{-2t},$$
$$D^2x = 4X_2 t\, e^{-2t} - 4X_2\, e^{-2t}.$$

Equation (9.56) becomes

$$4X_2 t\, e^{-2t} - 4X_2\, e^{-2t} - 8X_2 t\, e^{-2t} + 4X_2\, e^{-2t} + 2X_2 t\, e^{-2t} - 6Y_2 t\, e^{-2t} = 0,$$
$$-2X_2 t\, e^{-2t} - 6Y_2 t\, e^{-2t} = 0. \quad (9.78)$$

Equations (9.57) and (9.58) become

$$6Y_2 t\, e^{-2t} - 3Z_2 t\, e^{-2t} = 0, \quad (9.79)$$

$$X_2 t\, e^{-2t} + Y_2 t\, e^{-2t} + Z_2 t\, e^{-2t} = 0. \quad (9.80)$$

When we divide equations (9.78) to (9.80) by $t\, e^{-2t}$, we have

$$-2X_2 - 6Y_2 = 0, \quad (9.81)$$

$$6Y_2 - 3Z_2 = 0, \quad (9.82)$$

$$X_2 + Y_2 + Z_2 = 0. \quad (9.83)$$

The determinant of the coefficients

$$\begin{vmatrix} -2 & -6 & 0 \\ 0 & 6 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

is of rank 2. We can solve for X_2 and Z_2 in terms of Y_2 and obtain

$$X_2 = -3Y_2, \quad (9.84)$$

$$Z_2 = 2Y_2. \quad (9.85)$$

A solution of the equations is

$$x = -3Y_2 t\, e^{-2t}, \quad (9.86)$$

$$y = Y_2 t\, e^{-2t}, \quad (9.87)$$

$$z = 2Y_2 t\, e^{-2t}. \quad (9.88)$$

The general solution of the set of simultaneous equations is

$$x = -3Y_1\, e^{-2t} - 3Y_2 t\, e^{-2t}, \quad (9.89)$$

$$y = Y_1\, e^{-2t} + Y_2 t\, e^{-2t}, \quad (9.90)$$

$$z = 2Y_1\, e^{-2t} + 2Y_2 t\, e^{-2t}. \quad (9.91)$$

Y_1 and Y_2 are the two constants of integration required by a second order set of equations.

Example 2.

$$(D + 2)x - (2D + 4)y = 0. \tag{9.92}$$

$$(D + 2)x - (3D + 6)z = 0. \tag{9.93}$$

$$x + y + z = 0. \tag{9.94}$$

As before, substitute $x = X_1 e^{mt}$, $y = Y_1 e^{mt}$, $z = Z_1 e^{mt}$. We have, after dividing by e^{mt},

$$(m + 2)X_1 - (2m + 4)Y_1 = 0, \tag{9.95}$$

$$(m + 2)X_1 - (3m + 6)Z_1 = 0, \tag{9.96}$$

$$X_1 + Y_1 + Z_1 = 0. \tag{9.97}$$

The characteristic equation is obtained from the coefficients of equations (9.95) to (9.97) as

$$\begin{vmatrix} (m+2) & -(2m+4) & 0 \\ (m+2) & 0 & -(3m+6) \\ 1 & 1 & 1 \end{vmatrix} = 0. \tag{9.98}$$

On expansion of the determinant symbol, this equation becomes

$$m^2 + 4m + 4 = 0. \tag{9.99}$$

We have a second order set of equations and the characteristic equation has a double root -2. Substitute $x = X_1 e^{-2t}$, $y = Y_1 e^{-2t}$, $z = Z_1 e^{-2t}$ in equations (9.92) to (9.94) and we obtain

$$(-2 + 2)X_1 e^{-2t} - (-4 + 4)Y_1 e^{-2t} = 0, \tag{9.100}$$

$$(-2 + 2)X_1 e^{-2t} - (-6 + 6)Z_1 e^{-2t} = 0, \tag{9.101}$$

$$X_1 e^{-2t} + Y_1 e^{-2t} + Z_1 e^{-2t} = 0. \tag{9.102}$$

If these are divided by e^{-2t}, we have

$$0X_1 + 0Y_1 + 0Z_1 = 0, \tag{9.103}$$

$$0X_1 + 0Y_1 + 0Z_1 = 0, \tag{9.104}$$

$$X_1 + Y_1 + Z_1 = 0. \tag{9.105}$$

The determinant of the coefficients is

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

and is of rank one. We can solve for one unknown in terms of the other two.

$$X_1 = -(Y_1 + Z_1).$$

The general solution of the set of equations is

$$x = -(Y_1 + Z_1)\, e^{-2t}, \tag{9.106}$$

$$y = Y_1 e^{-2t}, \tag{9.107}$$

$$z = Z_1 e^{-2t}, \tag{9.108}$$

4. $(D^2 + 3D)y = \cosh 2x$
5. $(D^3 + 2D^2 - D - 2)y = e^{2x} + e^x + x \sin 2x$
6. $(D^3 + 5D^2)y = \sinh 3x$
7. $(D^4 + 3D^3 + 3D^2 + D)y = e^x + e^{-x} + x^2$
8. $(D^3 - 4D^2 + 3D + 2)y = e^{3x}$
9. $(D^3 - 2D^2 + 5D + 26)y = \sin 2x$

Find the general solution of the following sets of simultaneous equations.

10. $\begin{cases} \dfrac{d^2x}{dt^2} - 6x + 2y = 0 \\ \dfrac{d^2y}{dt^2} - 7y + 3x = 0 \end{cases}$

11. $\begin{cases} \dfrac{d^2x}{dt^2} - 6x + 2y = t \\ \dfrac{d^2y}{dt^2} - 7y + 3x = 2t + 4 \end{cases}$

12. $\begin{cases} \dfrac{d^2x}{dt^2} - 6x + 2y = e^t \\ \dfrac{d^2y}{dt^2} - 7y + 3x = \sin 3t \end{cases}$

13. $\begin{cases} \dfrac{dx}{dt} + 3x - 12y = t \\ \dfrac{dy}{dt} + 3y + 3x = \sin 2t \end{cases}$

CHAPTER 10

GAMMA FUNCTIONS AND BESSEL'S FUNCTIONS

10.1 Introduction. If a heavy chain is supported at one end and is permitted to swing as a pendulum, the motion cannot be expressed in simple functions like powers, exponentials, logarithms, and trigonometric functions. When a mathematical analysis is made, the following equation is obtained.

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + x^2 y = 0.$$

The same type of equation is obtained when problems are worked involving round wires, shafts, round diaphragms, and other objects that are round in section. The importance of the problem for round wires, etc., makes it worth while for the engineer to be able to solve the above equation.

The equation above is one form of Bessel's equation. Its solution is written formally in terms of Bessel's functions, and therefore we must determine what Bessel's functions are and how to use them. It is desirable, however, to study the factorial function and the gamma function before considering Bessel's functions.

10.2 Factorial Function. The factorial function $n!$ is defined, for a positive integer >1, as the continued product of the first n positive integers. Thus, $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$, etc. In general, we have

$$\begin{aligned} n! &= n(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1 \\ &= n[(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1] = n(n-1)! \end{aligned}$$

This relation enables us to define two special cases not covered by the definition given above. If we let $n = 2$ in

$$(n-1)! = \frac{n!}{n}$$

we have

$$1! = \frac{2 \cdot 1}{2} = 1.$$

Now, let $n = 1$ and we have

$$0! = \frac{1!}{1} = 1.$$

The numerical values of $n!$ are tabulated for a few values of n in Table X-1.

TABLE X-1

n	$n!$	n	$n!$
0	1	5	120
1	1	6	720
2	2	7	5040
3	6	8	40,320
4	24	9	362,880

10.3 Gamma Function. The gamma function is a sort of generalization of the factorial function. The gamma function $\Gamma(n)$ is defined for any real value of n except zero and negative integer values. If n is a positive integer we shall find that $\Gamma(n) = (n - 1)!$ It may seem unfortunate at first that the "minus one" appears in this equation, but it offers no real difficulty.

Definition. The gamma function is defined by the following definite integral:

$$\Gamma(n) = \int_0^\infty z^{n-1} e^{-z}\, dz \qquad n > 0. \tag{10.1}$$

For $n = 1$, we have $\Gamma(1) = \int_0^\infty e^{-z}\, dz = \Big[-e^{-z}\Big]_0^\infty = -0 + 1 = 1.$

10.4 In order to investigate the properties of the gamma function we shall first consider the integration by parts of the following integral:

$$\int_0^t z^{n-1} e^{-z}\, dz = \int_0^t u\, dv = uv\Big|_0^t - \int_0^t v\, du \tag{10.2}$$

Let $z^{n-1} = u$ and $e^{-z}\, dz = dv$; then $du = (n - 1)z^{n-2}\, dz$ and $v = -e^{-z}$.

$$\int_0^t z^{n-1} e^{-z}\, dz = -z^{n-1} e^{-z}\Big|_0^t - \int_0^t (-e^{-z})(n - 1)z^{n-2}\, dz \tag{10.3}$$

$$= -t^{n-1} e^{-t} + (n - 1)\int_0^t z^{n-2} e^{-z}\, dz \tag{10.3$'$}$$

We shall show that, as t becomes large, $t^{n-1} e^{-t}$ approaches zero. To do this we show that, as t is made large,

$$\frac{e^t}{t^{n-1}}$$

approaches infinity. MacLaurin's series for e^t is

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots + \frac{t^m}{m!} + \cdots. \tag{10.4}$$

For any value of $n - 1$, there is a term in the series

$$\frac{t^m}{m!}$$

such that $m > n - 1$. Therefore, we write

$$\frac{e^t}{t^{n-1}} = \frac{1}{t^{n-1}} + \frac{1}{t^{n-2}} + \frac{1}{2!t^{n-3}} + \cdots + \frac{t^{m-n+1}}{m!} + \cdots. \tag{10.5}$$

Since $m > n - 1$, $m - n + 1$ is larger than 0, and t with this quantity as an exponent

$$t^{m-n+1}$$

approaches infinity as t approaches infinity. Therefore, if we let t approach infinity in equation (10.3′), we have

$$\Gamma(n) = \int_0^\infty z^{n-1} e^{-z}\, dz = (n-1)\int_0^\infty z^{(n-1)-1} e^{-z}\, dz. \tag{10.6}$$

We recognize the second integral in equation (10.6) as $\Gamma(n-1)$. Therefore, we have shown that

$$\Gamma(n) = (n-1)\Gamma(n-1). \tag{10.7}$$

From equation (10.7), we have

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}. \tag{10.7'}$$

10.5 Equation 10.7 enables us to show the relation between the factorial function and the gamma function of a positive integer. Repeated use of equation (10.7) gives us, if n is a positive integer,

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2)\\ &= (n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1\cdot \Gamma(1) = (n-1)!\,\Gamma(1).\end{aligned}$$

Since it was shown in section 10.3 that $\Gamma(1) = 1$, we have $\Gamma(n) = (n-1)!$ if n is a positive integer.

Values for $\Gamma(n)$, for values of n between 1 and 2, will be found in Dwight's "Tables of Integrals and other Mathematical Data," Macmillan Company and values of the logarithms of $\Gamma(n)$ will be found in Peirce's "A Short Table of Integrals," Ginn and Company.

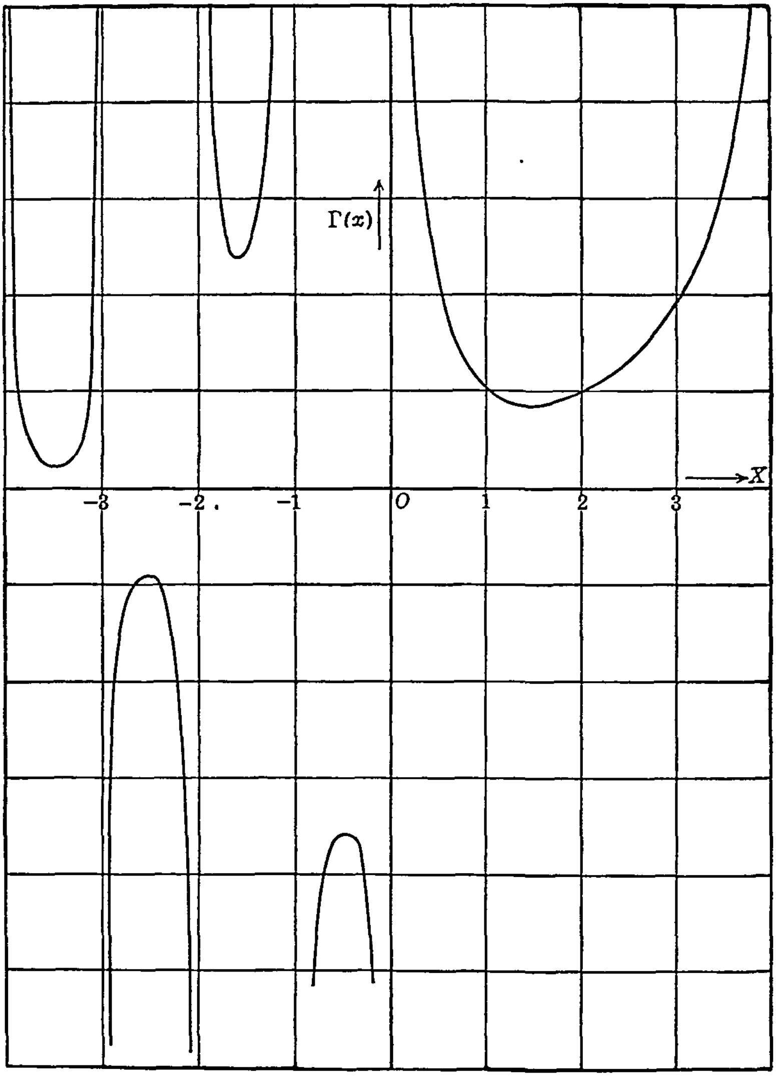

FIG. 10–1

Knowing $\Gamma(n)$ for values of n between 1 and 2, we can compute $\Gamma(n)$ for any positive value of n with the aid of equation (10.7) or (10.7′).

$$\begin{aligned}\Gamma(3.2) = 2.2\Gamma(2.2) &= (2.2)(1.2)\Gamma(1.2)\\ &= (2.2)(1.2)(0.9182) = 2.424.\end{aligned}$$

$$\Gamma(0.6) = \frac{\Gamma(1.6)}{0.6} = \frac{0.8935}{0.6} = 1.489.$$

We can extend the definition to include negative values of n, except negative integers. Equation (10.7′) gives

$$\Gamma(-0.4) = \frac{\Gamma(0.6)}{-0.4} = \frac{\Gamma(1.6)}{(-0.4)(0.6)} = -3.723.$$

The curve in Fig. 10–1 shows the variation of the gamma function.

10.6 The numerical value of $\Gamma(0.5)$ can be determined by the following device which depends upon the fact that a definite integral is not a function of the variable of integration. The definite integral

$$\int_0^2 x\,dx = 2$$

has the same value whether we write it any of the following ways:

$$\int_0^2 y\,dy = 2,\quad \int_0^2 z\,dz = 2,\quad \int_0^2 w\,dw = 2,\quad \int_0^2 \phi\,d\phi = 2.$$

$\Gamma(0.5)$ can be written using the definition equation (10.1)

$$\Gamma(0.5) = \int_0^\infty z^{-0.5}\,e^{-z}\,dz. \tag{10.8}$$

We can eliminate the fractional exponent by substituting $z = x^2$; then $dz = 2x\,dx$ and (10.8) becomes

$$\Gamma(0.5) = \int_0^\infty x^{-1}\,e^{-x^2}\,2x\,dx = 2\int_0^\infty e^{-x^2}\,dx. \tag{10.9}$$

We have the same limits since, when $z = 0$, $x = 0$, and when $z = \infty$, $x = \infty$. We now change the variable of integration in the last integral.

$$\Gamma(0.5) = 2\int_0^\infty e^{-y^2}\,dy. \tag{10.10}$$

If we multiply (10.9) and (10.10), we have

$$[\Gamma(0.5)]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy. \tag{10.11}$$

This is a surface integral over the area in the first quadrant between the x and y axes. If we change to polar coordinates r, θ, $x^2 + y^2 = r^2$ and the element of area $dy\,dx$ becomes $rd\theta dr$. Therefore

$$[\Gamma(0.5)]^2 = 4\int_0^\infty \int_0^{\pi/2} e^{-r^2}\, r\, d\theta\, dr \tag{10.12}$$

$$= 2\pi \int_0^\infty e^{-r^2} r\, dr = \pi(-0 + 1) = \pi$$

and we have $\Gamma(0.5) = \sqrt{\pi}$. (10.13)

10.7 Bessel's Equation. Many problems involving circular constructions, such as round diaphragms, wires, pipes and tubes, horns, etc., lead to a differential equation of the following form:

$$z^2 \frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2 - n^2)y = 0 \tag{10.14}$$

where n is a number whose value is determined by the particular problem. This type of equation is known as Bessel's equation since the first intensive study of its solution was made by Bessel. Bessel's equation is a second order homogeneous linear differential equation; therefore, its general solution contains two constants of integration. If y_1 and y_2 are independent solutions, $y = C_1y_1 + C_2y_2$ is the general solution where C_1 and C_2 are arbitrary constants.

10.8 Bessel's Function. Let $y_1 = J(z)$ be a particular solution of equation (10.14). The form of $J(z)$ depends on the value of n in the given equation. This dependence of the form of $J(z)$ on the value of n is indicated by using a subscript on the functional letter J. If $n = 0$ in the given equation, we write a particular solution $y_1 = J_0(z)$. If $n = 2$ in the given equation, a particular solution is $y_1 = J_2(z)$. In the general case, a particular solution of equation (10.14) is written $y_1 = J_n(z)$. This function is called Bessel's function of the first kind of order n. The reason it is called the function of the first kind will become evident later.

A formula is derived in section 10.10 for $J_n(z)$ in terms of n and z. Values for $J_n(z)$ have been computed by mathematicians for certain values of n and are tabulated just as values of $\sin x$ and $\log x$ are listed in trigonometric tables and tables of logarithms. A few values of $J_0(x)$ and $J_1(x)$ are to be found in Table X-2. For a more complete table the student is referred to Jahnke-Emde "Tables of Functions."

10.9 Another Form of Bessel's Equation. Let $z = ax$. Then

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{1}{a}\frac{dy}{dx},$$

$$\frac{d^2y}{dz^2} = \frac{1}{a^2}\frac{d^2y}{dx^2}$$

Substituting in equation (10.14), we have

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (a^2x^2 - n^2)y = 0. \tag{10.15}$$

If $y_1 = J_n(z)$ is a solution of equation (10.14), $y_1 = J_n(ax)$ is a solution of equation (10.15). Equation (10.14) can be considered as a special case of equation (10.15).

10.10 A Solution of Bessel's Equation. In order to derive an expression for y_1 in terms of z to satisfy equation (10.14), we make the following substitution:

$$y_1 = a_0z^r + a_1z^{r+1} + a_2z^{r+2} + \cdots \quad \text{where } a_0 \neq 0 \tag{10.16}$$

and find what values of r, a_0, a_1, etc., will make this expression for y_1 satisfy the equation.

Differentiating (10.16), we have

$$\frac{dy_1}{dz} = ra_0z^{r-1} + (r+1)a_1z^r + (r+2)a_2z^{r+1} + \cdots, \tag{10.17}$$

$$\frac{d^2y_1}{dz^2} = r(r-1)a_0z^{r-2} + (r+1)ra_1z^{r-1} + (r+2)(r+1)a_2z^r + \cdots. \tag{10.18}$$

Substituting these expressions into (10.14), we obtain a power series

$$A_0z^r + A_1z^{r+1} + A_2z^{r+2} + \cdots \tag{10.19}$$

where

$$\begin{aligned}
A_0 &= a_0r(r-1) + a_0r - n^2a_0\\
A_1 &= a_1(r+1)r + a_1(r+1) - n^2a_1\\
A_2 &= a_2(r+2)(r+1) + a_2(r+2) + a_0 - n^2a_2\\
A_3 &= a_3(r+3)(r+2) + a_3(r+3) + a_1 - n^2a_3\\
A_4 &= a_4(r+4)(r+3) + a_4(r+4) + a_2 - n^2a_4\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
A_m &= a_m(r+m)(r+m-1) + a_m(r+m) + a_{m-2} - n^2a_m
\end{aligned}$$

These expressions simplify to

$$
\begin{aligned}
A_0 &= a_0(r^2 - n^2) \\
A_1 &= a_1[(r+1)^2 - n^2] \\
A_2 &= a_2[(r+2)^2 - n^2] + a_0 \\
A_3 &= a_3[(r+3)^2 - n^2] + a_1 \\
A_4 &= a_4[(r+4)^2 - n^2] + a_2 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
A_m &= a_m[(r+m)^2 - n^2] + a_{m-2}
\end{aligned}
$$

If y_1 is a solution of equation (10.14), the power series (10.19) equals zero for any value of z whatsoever. Therefore, each coefficient equals zero, or $A_0 = A_1 = A_2 = \cdots = 0$.

If $A_0 = a_0(r^2 - n^2) = 0$, while $a_0 \neq 0$, then $r^2 = n^2$. Then, since $A_1 = a_1(2r + 1) = 0$, $a_1 = 0$ except for the special case where $n^2 = 0.25$ so that r can be equal to -0.5. We shall consider the general case now and therefore make $a_1 = 0$. At first this seems as though we are neglecting a possible solution. In the following section we consider the special case where $n^2 = 0.25$ to show that we lose nothing by assuming in general that $a_1 = 0$.

$$A_2 = a_2[r^2 + 4r + 4 - n^2] + a_0 = 0.$$

$$a_2 = -\frac{a_0}{4(r+1)}.$$

$$A_3 = a_3(r^2 + 6r + 9 - n^2) + a_1 = 0.$$

If $a_1 = 0$, $a_3 = 0$ unless $r = -1.5$. This special case is included in the general case if we make $a_3 = 0$. (See the following section.) In general all the a's with odd subscripts are made zero.

$$A_4 = a_4(r^2 + 8r + 16 - n^2) + a_2 = 0$$

$$a_4 = -\frac{a_2}{8(r+2)} = \frac{a_0}{4 \cdot 8(r+1)(r+2)}$$

$$A_6 = a_6(r^2 + 12r + 36 - n^2) + a_4 = 0$$

$$a_6 = -\frac{a_4}{12(r+3)} = \frac{-a_0}{4 \cdot 8 \cdot 12(r+1)(r+2)(r+3)}$$

$$= \frac{(-1)^3 a_0}{2^6 3!(r+1)(r+2)(r+3)}.$$

The general formula for the a's with even subscripts is

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k!(r+1)(r+2)\cdots(r+k)} \tag{10.20}$$

and a_0 is arbitrary. Since each coefficient is proportional to a_0, the solution y_1 is proportional to a_0. Now, since y_1 times any constant is a solution (Bessel's equation is a linear homogeneous differential equation), we can assign to a_0 any value that we find convenient. It is customary to let $a_0 = \dfrac{1}{2^r\Gamma(r+1)}$. Then, equation (10.20) becomes

$$a_{2k} = \frac{(-1)^k}{2^{r+2k}k!\,\Gamma(r+k+1)}. \tag{10.21}$$

Since $r^2 = n^2$, we can have $r = n$ or $-n$. If $r = n$, not an integer, we have

$$y_1 = J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{2^{n+2k}k!\,\Gamma(n+k+1)}. \tag{10.22}$$

If we let $r = -n$, not an integer, we have

$$y_2 = J_{-n}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{-n+2k}}{2^{-n+2k}k!\,\Gamma(-n+k+1)}. \tag{10.23}$$

y_1 and y_2 as defined in equations (10.22) and (10.23) will each satisfy equation (10.14). Furthermore, y_1 is not a multiple of y_2 when n is not an integer (the first term in y_1 contains z^n while the first term in y_2 contains z^{-n}). Therefore

$$y = C_1y_1 + C_2y_2 = C_1J_n(z) + C_2J_{-n}(z) \tag{10.24}$$

is the general solution of equation (10.14).

If n is zero or a positive integer, we can write equation (10.22) as

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{n+2k}}{2^{n+2k}k!\,(n+k)!}. \tag{10.25}$$

Since $(k-n)!$ is not defined for $k < n$, equation (10.23) is meaningless if we make n a positive integer. However, as n approaches a positive integer N, the first N terms approach zero. We omit the first n terms, which are the only terms which would contain the factorial of a negative integer, and when n is a positive integer define $J_{-n}(z)$ as

$$J_{-n}(z) = \sum_{k=n}^{\infty} \frac{(-1)^k z^{-n+2k}}{2^{-n+2k}k!\,(-n+k)!}. \tag{10.26}$$

Let $k = n + s$

$$J_{-n}(z) = \sum_{s=0}^{\infty} \frac{(-1)^s(-1)^n z^{n+2s}}{2^{n+2s}(n+s)!\,s!} = (-1)^n J_n(z). \tag{10.27}$$

Therefore, if n is an integer, $J_n(z)$ and $J_{-n}(z)$ are not independent solutions, and some other device must be employed to obtain the general solution.

If n is an integer, the general solution is written

$$y = C_1 J_n(z) + C_2 N_n(z). \tag{10.28}$$

$N_n(z)$ is known as Bessel's function of order n of the second kind. We shall not derive an expression for $N_n(z)$ at this time.

10.11 Special Case, $n^2 + 0.25$. We shall now consider the special case where $n^2 = 0.25$. In this case if $r = -0.5$ a_1 need not be zero and we have

$$A_3 = a_3(6r + 9) + a_1 = 0,$$

$$a_3 = \frac{-a_1}{3(2r+3)} = \frac{-a_1}{6} = \frac{-a_1}{3!},$$

$$A_5 = a_5(10r + 25) + a_3 = 0,$$

$$a_5 = \frac{-a_3}{5(2r+5)} = \frac{-a_3}{5 \cdot 4} = \frac{a_1}{5!},$$

$$A_7 = a_7(14r + 49) + a_5 = 0,$$

$$a_7 = \frac{-a_5}{7(2r+7)} = \frac{-a_5}{7 \cdot 6} = \frac{-a_1}{7!}.$$

We also have, when $r = -0.5$,

$$a_2 = \frac{-a_0}{2},$$

$$a_4 = \frac{-a_2}{3 \cdot 4} = \frac{a_0}{4!},$$

$$a_6 = \frac{-a_4}{5 \cdot 6} = \frac{-a_0}{6!}.$$

Therefore, for our general solution in this case, we have

$$y_1 = a_0\left[z^{-0.5} - \frac{z^{1.5}}{2} + \frac{z^{3.5}}{4!} - \frac{z^{5.5}}{6!} + \cdots\right]$$
$$+ a_1\left[z^{0.5} - \frac{z^{2.5}}{3!} + \frac{z^{4.5}}{5!} - \frac{z^{6.5}}{7!} + \cdots\right] \tag{10.29}$$

containing two arbitrary constants. If we now let $r = 0.5$, we have

$$a_2 = \frac{-a_0}{2 \cdot 3} = \frac{-a_0}{3!},$$

$$a_4 = \frac{-a_2}{4 \cdot 5} = \frac{a_0}{5!},$$

$$a_6 = \frac{-a_4}{6 \cdot 7} = \frac{-a_0}{7!}.$$

This gives for a solution

$$y_2 = a_0 \left[z^{0.5} - \frac{z^{2.5}}{3!} + \frac{z^{4.5}}{5!} - \frac{z^{6.5}}{7!} + \cdots \right]. \tag{10.30}$$

Although it seemed as though we had lost the second part of (10.29) by making $a_1 = 0$, it is now evident that we have it in (10.30).

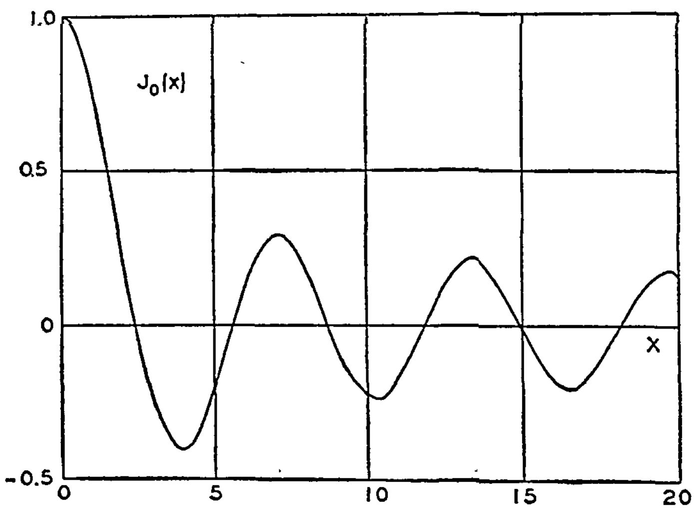

Fig. 10–2

10.12 Bessel's Function of the First Kind of Order Zero. $J_0(z)$ is quite frequently met in engineering problems. In this case (10.25) becomes

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{2^{2k}(k!)^2} \tag{10.31}$$

$$= 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} + \frac{z^8}{2^2 4^2 6^2 8^2} - \cdots.$$

The graph of this function for real values of z is like a cosine curve which tapers off as it goes from $z = 0$; see Fig. 10–2. It is shown in a later

chapter that a circular drum head or diaphragm can vibrate so that a section through the center of the circle would have displacements proportional to the ordinates of the curve for $J_0(z)$.

TABLE X-2

x	ber x	bei x	ber′ x	bei′ x	$J_0(x)$	$J_1(x)$	x
0 0	1 0000	0 0000	0 0000	0 0000	1 0000	0 0000	0 0
0 5	0 9990	0 0625	−0 0078	0 2499	0 9385	0 2423	0 5
1 0	0 9844	0 2496	−0 0624	0 4974	0 7652	0 4401	1 0
1 5	0 9211	0 5576	−0 2100	0 7302	0 5118	0 5579	1 5
2 0	0 7517	0 9723	−0 4931	0 9170	0 2239	0 5767	2 0
2 5	0 4000	1 4572	−0 9436	0 9983	−0 0484	0 4971	2 5
3 0	−0 2214	1 9376	−1 570	0 8805	−0 2601	0 3391	3 0
3 5	−1 194	2 2832	−2 336	0 4353	−0 3801	0 1374	3 5
4 0	−2 563	2 2927	−3 135	−0 491	−0 3971	−0 0660	4 0
4 5	−4 299	1 686	−3 754	−2 053	−0 3205	−0 2311	4 5
5 0	−6 230	0 116	−3 845	−4 354	−0 1776	−0 3276	5 0
5 5	−7 974	−2 789	−2 907	−7 373	−0 0068	−0 3414	5 5
6 0	−8 858	−7 335	−0 293	−10 846	0 1506	−0 2767	6 0
6 5	−7 867	−13 607	4 717	−14 129	0 2601	−0 1538	6 5
7 0	−3 633	−21 239	12 765	−16 041	0 3001	−0 0047	7 0
7 5	5 455	−29 116	24 130	−14 736	0 2663	0 1352	7 5
8 0	20 974	−35 02	38 31	−7 680	0 1717	0 2346	8 0
8 5	43 94	−35 30	53 44	8 290	0 0419	0 2731	8.5
9 0	73 94	−24 71	65 60	36 30	−0 0903	0.2435	9 0
9 5	107 95	3 41	68 13	78 68	−0 1939	0 1613	9 5
10 0	138 84	56 37	51 20	135 31	−0 2459	0 0435	10 0

10.13 Zeros of Bessel's Function. The points where the curve in Fig. 10–2 passes through zero, i.e., the value of x for which $J_0(x) = 0$, are very important in studying motions of diaphragms. Table X–3 gives the first twenty roots of $J_0(x) = 0$. It will be noticed that the difference between the successive values of the roots is very nearly equal to π for the large roots.

TABLE X–3

First twenty roots of $J_0(x) = 0$			
2 40	18 07	33 78	49 48
5 52	21 21	36 92	52 62
8 65	24 35	40 06	55 77
11 79	27 49	43 20	58 91
14 93	30 63	46 34	62 05

CHAPTER 12

VECTOR CALCULUS

12.1 Point Functions and Fields. If the air temperature is measured by several people at the same time we would expect some of the observed temperatures to be different from others. The air temperature at an instant of time is a function of the position of the point of interest; the temperature at any point depends upon the location of the point. The temperature, a scalar, is a scalar point function or a scalar field.

If the observers were measuring wind velocity instead of temperature they would note it in miles per hour in a definite direction. The wind velocity, a vector, is at each point a function of the location of the point and therefore the velocity is a vector point function or a vector field.

12.2 Derivatives. A scalar point function can be represented as a function of the coordinates of the point and perhaps time; the temperature of the air depends on the latitude, longitude, altitude, and time and is a function of four variables

$$\theta = f(x,y,z,t). \tag{12.1}$$

θ is the temperature at a point, x is the latitude, y the longitude, z the altitude of the point, and t is the time. θ can be differentiated with respect to any of the variables as is done for any type of scalar quantity.

When a vector point function is differentiated the result is defined as follows: Let the vector $\mathbf{R}$ be a function of location and time. Let $\mathbf{R} = \mathbf{R}_A$ be the value of the vector at point P and at time t_A, and let $\mathbf{R} = \mathbf{R}_B = \mathbf{R}_A + \Delta\mathbf{R}$ be the value of the vector at the same point P at a later instant of time $t_A + \Delta t$. See Fig. 12-1. If the vector $\Delta\mathbf{R}$ is divided by Δt and the ratio is found to approach a finite limit as Δt approaches zero, this limit is by definition the derivative of the vector point function $\mathbf{R}$ with respect to t at the point P. The symbol for the derivative is the same for the derivative of a vector field as for scalar derivatives.

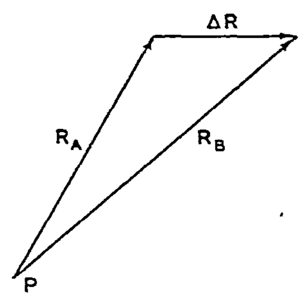

FIG. 12-1

$$\frac{d\mathbf{R}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{R}}{\Delta t}. \tag{12.2}$$

If we substitute $\mathbf{R} = R\mathbf{R}_1$, we have

$$\frac{d\mathbf{R}}{dt} = \frac{d(R\mathbf{R}_1)}{dt} = R\frac{d\mathbf{R}_1}{dt} + \mathbf{R}_1\frac{dR}{dt}. \tag{12.3}$$

If the magnitude of the vector $\mathbf{R}$ is constant, R is constant and

$$\frac{dR}{dt} = 0$$

and equation (12.3) becomes in this case

$$\frac{d\mathbf{R}}{dt} = R\frac{d\mathbf{R}_1}{dt}. \qquad (R \text{ constant}) \tag{12.4}$$

Referring to Fig. 12–2, if R is constant, the triangle is an isosceles triangle and as Δt approaches zero $\Delta\mathbf{R}$ approaches zero, and the angle θ in Fig. 12–2 approaches 90°. Therefore, in the limit we find the derivative of $\mathbf{R}$ is perpendicular to $\mathbf{R}$ if R is constant.

$\mathbf{R}_1$ is a vector of constant length (unity). Therefore the derivative of $\mathbf{R}_1$ is perpendicular to $\mathbf{R}_1$ and also perpendicular to $\mathbf{R}$. Since

$$\mathbf{R}_1\frac{dR}{dt}$$

is parallel to $\mathbf{R}$, we find that the two vectors on the right of equation (12.3) are perpendicular to each other. The first vector is perpendicular to $\mathbf{R}$ and the second one parallel to $\mathbf{R}$.

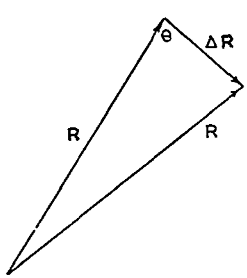

FIG. 12–2

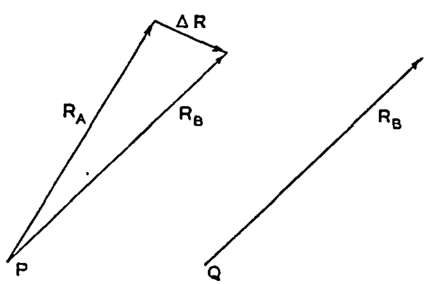

FIG. 12–3

If the vector point function $\mathbf{R}$, which depends on three space coordinates x, y, and z, as well as time t, is differentiated with respect to x instead of t, the results obtained above when we differentiated with respect to t still hold true. In Fig. 12–3 the value of $\mathbf{R}$ at the point P is $\mathbf{R}_A$, the value at Q at the same instant of time is $\mathbf{R}_B$, and the distance between P and Q is Δx. $\Delta\mathbf{R}$ is found by drawing $\mathbf{R}_B$ from the same point from which $\mathbf{R}_A$ is

drawn and the figure obtained is just the same as we had before. The derivative is found by finding the limit if it exists of $\Delta\mathbf{R}$ divided by Δx when Δx approaches zero.

$$\frac{d\mathbf{R}}{dx} = \lim_{\Delta x \to 0} \frac{\Delta \mathbf{R}}{\Delta x}. \tag{12.5}$$

12.3 Integration. Integration of vector functions is analogous to integration of scalar functions. The line integral

$$\int_A^B \mathbf{F}\, dS$$

is defined as follows: Divide the path A to B (Fig. 12-4) into a number of segments, the longest one being ΔS. Let $\mathbf{F}_1$ be the vector function

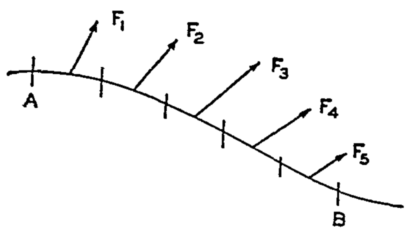

FIG. 12-4

somewhere in the first interval ΔS_1, let $\mathbf{F}_2$ be the vector function somewhere in the second segment ΔS_2, etc. The sum of the products of the vectors in each segment multiplied by the length of the respective segment is

$$\sum_{AB} \mathbf{F}_p\, \Delta S_p.$$

If this sum has a limit as ΔS approaches zero, this limit is by definition the integral desired.

$$\int_A^B \mathbf{F}\, dS = \lim_{\Delta S \to 0} \sum_{AB} \mathbf{F}_p\, \Delta S_p. \tag{12.6}$$

Surface integrals and volume integrals are defined by analogous definitions. The elements are elements of area or volume rather than elements of length as for line integrals.

12.4 Gradient. Let θ be the temperature at any point in a body. If we set up x, y, and z coordinate axes, θ is a function of x, y, and z. Heat is conducted from points of higher temperature to points of lower temperature and if we are studying the heat flow in a body we must know how the temperature varies from point to point. We must consider the space

rate of change of the temperature, or temperature gradient. Temperature gradient tells how the temperature varies in degrees Centigrade per centimeter and in what direction the heat flow would be. The rate of temperature change along the x axis is $\partial\theta/\partial x$, the rate of change in the y direction is $\partial\theta/\partial y$ and along the z axis is $\partial\theta/\partial z$. These partial derivatives are the three components of the temperature gradient $\mathbf{P}$, and

$$\mathbf{P} = \mathbf{i}\frac{\partial\theta}{\partial x} + \mathbf{j}\frac{\partial\theta}{\partial y} + \mathbf{k}\frac{\partial\theta}{\partial z}.$$

$\mathbf{P}$ is the gradient of θ, and this is sometimes written

$$\mathbf{P} = \text{grad } \theta.$$

A useful abbreviation is arrived at as follows: Let the symbol ∇ (called " del ") represent an operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

Then

$$\nabla\theta = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\theta = \mathbf{i}\frac{\partial\theta}{\partial x} + \mathbf{j}\frac{\partial\theta}{\partial y} + \mathbf{k}\frac{\partial\theta}{\partial z}.$$

The distributive law holds for finding the gradient.

$$\nabla(\theta + \phi) = \nabla\theta + \nabla\phi.$$

The commutative law fails. $\nabla\theta \neq \theta\nabla$ because $\theta\nabla$ is meaningless. The associative law becomes the rule for differentiating a product: $\nabla(\theta\phi) = \phi\nabla\theta + \theta\nabla\phi$.

As an example of finding the gradient consider a scalar field θ, such that θ at each point equals the distance from the origin to the point, or $\theta = r = \sqrt{x^2 + y^2 + z^2}$. To find grad θ we proceed as follows:

$$\begin{aligned} \text{grad } \theta = \nabla\theta &= \mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z} \\ &= \mathbf{i}\frac{x}{r} + \mathbf{j}\frac{y}{r} + \mathbf{k}\frac{z}{r}. \end{aligned}$$

If $\mathbf{r}$ is the radius vector of a point, that is, $\mathbf{r}$ tells both the distance of a point from the origin and the direction as well,

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z,$$

$$\nabla r = \mathbf{i}\frac{x}{r} + \mathbf{j}\frac{y}{r} + \mathbf{k}\frac{z}{r} = \mathbf{r}_1.$$

It is suggested as an exercise for the student to show that

$$\nabla r^n = nr^{n-1}\mathbf{r}_1.$$

12.5 Divergence. Consider the motion of the air at a point P with coordinates x, y, and z in space. The velocity V of the air has components V_x, V_y, and V_z parallel to the three coordinate axes. Consider a rectangular prism $\Delta x \Delta y \Delta z$ with the point P at the center. See Fig. 12–5. The upward velocity of the air at the center of the bottom face of $\Delta x \Delta y \Delta z$ is

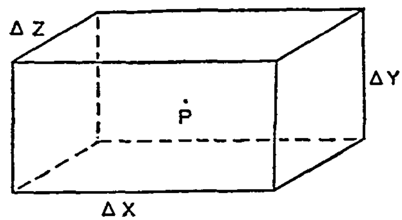

FIG. 12–5

$$V_y - \frac{\partial V_y}{\partial y}\frac{\Delta y}{2}.$$

With Δx and Δz small there is little error in assuming that this is the average upward velocity over the face of the prism. The upward velocity on the upper face of the prism is similarly found to be

$$V_y + \frac{\partial V_y}{\partial y}\frac{\Delta y}{2}.$$

The total volume of air passing through these faces is found by multiplying the average velocity expressed above by the area of the face $\Delta x \Delta z$. Following this procedure the total volume of air leaving the prism is found to be

$$\left(V_x + \frac{\partial V_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z + \left(V_y + \frac{\partial V_y}{\partial y}\frac{\Delta y}{2}\right)\Delta x \Delta z$$
$$+ \left(V_z + \frac{\partial V_z}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y.$$

The total air entering the prism is

$$\left(V_x - \frac{\partial V_x}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z + \left(V_y - \frac{\partial V_y}{\partial y}\frac{\Delta y}{2}\right)\Delta x \Delta z$$
$$+ \left(V_z - \frac{\partial V_z}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y.$$

The total volume of the air leaving, minus the air entering, is

$$\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)\Delta x \Delta y \Delta z.$$

If this is divided by the volume $\Delta x \Delta y \Delta z$ and the volume is allowed to shrink to zero in all directions, we have

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

as the increase in the volume of air per unit volume. This is by definition the **divergence** of the air velocity **V** and is sometimes written div **V**. The divergence of **V** can also be written $\nabla \cdot \mathbf{V}$ since

$$\nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}V_x + \mathbf{j}V_y + \mathbf{k}V_z) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.$$

The divergence obeys the distributive law.

$$\nabla \cdot (\mathbf{V} + \mathbf{U}) = \nabla \cdot \mathbf{V} + \nabla \cdot \mathbf{U}$$

but of course does not obey the cummutative law, $\nabla \cdot \mathbf{V} \neq \mathbf{V} \cdot \nabla$, since $\mathbf{V} \cdot \nabla$ is meaningless.

An interesting example of divergence is the case of the vector field **r** where the magnitude is everywhere equal to the distance to the origin, and the direction is away from the origin.

$$\nabla \cdot \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = 3.$$

12.6 Gauss' Theorem or the Divergence Theorem. The divergence theorem states mathematically that the total flux leaving a body equals the flux that originates inside the body. The divergence is the flux originating per unit volume; therefore, $(\nabla \cdot \mathbf{V})\, dv$ is the flux originating in the elementary volume dv. The integral of this throughout the volume of the body

$$\int \nabla \cdot \mathbf{V}\, dv$$

is the total flux originating inside the body. Let **n** be a unit vector normal to the surface of the body at point P. Then $\mathbf{V} \cdot \mathbf{n}$ is the normal component of the flux per unit area and $\mathbf{V} \cdot \mathbf{n}\, da$ or $\mathbf{V} \cdot d\mathbf{a}$ is the normal flux leaving through the elementary area da. Integrating over the entire surface, we have

$$\int \mathbf{V} \cdot d\mathbf{a}$$

as the total flux leaving the body. Therefore

$$\int \nabla \cdot \mathbf{V}\, dv = \int \mathbf{V} \cdot d\mathbf{a}.$$

This is a useful formula enabling us to replace a volume integral by a surface integral and vice versa.

12.7 Curl. Consider a magnetic field having components H_x, H_y, and H_z at the point P, whose coordinates are x, y, z. Figure 12–6 shows an elementary rectangle $\Delta x \Delta y$ on a plane through P perpendicular to the z axis. The point P lies at the center of the rectangle $\Delta x \Delta y$. The average value of the horizontal component of the magnetic field along AB in Fig. 12–6 is

$$H_x - \frac{\partial H_x}{\partial y}\frac{\Delta y}{2}.$$

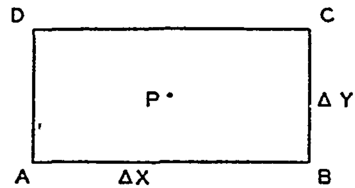

FIG. 12–6

If a unit magnetic pole is moved from A to B the work done by the field on the pole is

$$\left(H_x - \frac{\partial H_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x.$$

When the pole is moved from B to C the work done by the field on the pole is

$$\left(H_y + \frac{\partial H_y}{\partial x}\frac{\Delta x}{2}\right)\Delta y.$$

When the pole is moved from C to D the work done by the field on the pole is

$$-\left(H_x + \frac{\partial H_x}{\partial y}\frac{\Delta y}{2}\right)\Delta x.$$

And when the pole is moved from D to A the work done on the pole by the field is

$$-\left(H_y - \frac{\partial H_y}{\partial x}\frac{\Delta x}{2}\right)\Delta y.$$

The total work done by the field on the pole during the trip around the rectangle $\Delta x \Delta y$ is

$$\left(-\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x}\right)\Delta x \Delta y.$$

If this is divided by the area $\Delta x \Delta y$ and the area allowed to shrink to zero in all directions, we have

$$-\frac{\partial H_x}{\partial y} + \frac{\partial H_y}{\partial x}$$

as the work done per unit area by the field on a unit pole when the pole moves around an elementary area normal to the z axis.

To get a better picture of what we have, consider two unit N poles attached to a short bar with a pivot halfway between them. If the pivot is held parallel to the z axis, there will be a tendency for the poles to revolve about the pivot, proportional to

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}.$$

If the pivot is held parallel to the x axis the tendency for rotation can be found by a cyclic change of the letters x, y, z and is

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}.$$

The tendency for rotation about the y axis is proportional to

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}.$$

These three quantities tending to produce rotation about the x, y, and z axes are the x, y, and z components of the tendency to produce rotation about some single axis. That is, the vector

$$\mathbf{P} = \mathbf{i}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

measures the tendency to produce rotation and indicates the axis about which the rotation would take place. The vector **P** is defined as the curl of the vector **H**. **P** = curl **H**. The right-hand side of the above equation can be recognized as $\nabla \times \mathbf{H}$ and we have

$$\mathbf{P} = \text{curl}\,\mathbf{H} = \nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}.$$

It is suggested as an exercise for the student to show that curl $\mathbf{r} = 0$, where $\mathbf{r}$ is a vector field as described above in the discussion of divergence.

12.8 Stokes' Theorem. If $\nabla \times \mathbf{H}$ is proportional to the tendency to produce rotation about a certain axis, $\mathbf{n}\cdot(\nabla \times \mathbf{H})$ is proportional to the tendency to produce rotation about an axis parallel to the unit vector $\mathbf{n}$. Now $\mathbf{n}\cdot(\nabla \times \mathbf{H})\, da$ or $(\nabla \times \mathbf{H})\cdot d\mathbf{a}$ is the work done by the magnetic field on a unit pole when it is moved around an elementary area da normal to $\mathbf{n}$. Now consider any area, A, bounded by a curve, C, Fig. 12–7. Draw lines cutting the surface A into many small areas. If a unit pole is carried

around every area in the same direction, during the process the pole will be carried both ways on every path other than parts of the curve C. Therefore, the work done on all paths except C will cancel out. The work done will be the work done in going around the closed path C, or

$$\oint \mathbf{H} \cdot d\mathbf{s}.$$

The circle on the integral sign indicates that the integration is about a closed path.

Since the work done in moving the unit pole about the area $d\mathbf{a}$ is $(\nabla \times \mathbf{H}) \cdot d\mathbf{a}$, the sum of such expressions for all the areas of the surface A is

$$\sum (\nabla \times \mathbf{H}) \cdot d\mathbf{a}.$$

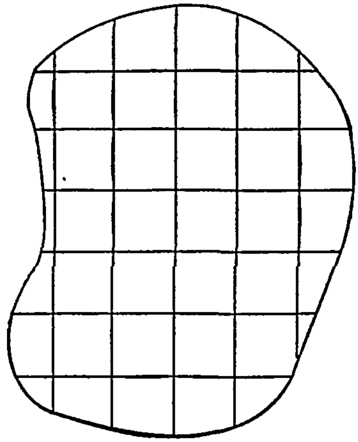

FIG. 12–7

If the number of dividing lines is increased so that every $d\mathbf{a}$ approaches zero the sum becomes an integral and we have

$$\int (\nabla \times \mathbf{H}) \cdot d\mathbf{a} = \oint \mathbf{H} \cdot d\mathbf{s}.$$

This is known as Stokes' theorem. It enables us to replace a surface integral by a line integral.

12.9 Repeated Use of Operator ∇. The operator ∇ can be used more than once, for example, the gradient of a scalar field is a vector field which may have a divergence. The number of possibilities where ∇ is used twice are

1. $\nabla \cdot (\nabla \theta) = \text{div grad } \theta.$
2. $\nabla \times (\nabla \theta) = \text{curl grad } \theta.$
3. $\nabla (\nabla \cdot \mathbf{V}) = \text{grad div } \mathbf{V}.$
4. $\nabla \cdot (\nabla \times \mathbf{V}) = \text{div curl } \mathbf{V}.$
5. $\nabla \times (\nabla \times \mathbf{V}) = \text{curl curl } \mathbf{V}.$

$$\nabla \cdot \nabla \theta = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2} = \nabla^2 \theta.$$

This is an important expression that occurs frequently in studies of electric and magnetic fields and elasticity. The operator $\nabla \cdot \nabla = \nabla^2$ is called the Laplacian.

$\nabla \times (\nabla\theta)$. Since the operator ∇ can be treated as an algebraic quantity so long as the order of factors is not disturbed, we can use the associative law to show $\nabla \times (\nabla\theta) = (\nabla \times \nabla)\theta = 0$.

$$\nabla(\nabla\cdot\mathbf{V}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right)$$

$$= \mathbf{i}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x \partial y} + \frac{\partial^2 V_z}{\partial x \partial z}\right) + \mathbf{j}\left(\frac{\partial^2 V_x}{\partial x \partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y \partial z}\right)$$

$$+ \mathbf{k}\left(\frac{\partial^2 V_x}{\partial x \partial z} + \frac{\partial^2 V_y}{\partial y \partial z} + \frac{\partial^2 V_z}{\partial z^2}\right).$$

This expression cannot be simplified further.

$\nabla\cdot(\nabla \times \mathbf{V})$. Since ∇ can be treated as an algebraic quantity as long as we do not disturb the order of the factors we can interchange the dot and cross and we have

$$\nabla\cdot(\nabla \times \mathbf{V}) = (\nabla \times \nabla)\cdot\mathbf{V} = 0.$$

$\nabla \times (\nabla \times \mathbf{V})$. We can use the formula for the triple product on this and we have

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla\cdot\mathbf{V}) - (\nabla\cdot\nabla)\mathbf{V}.$$

The term $\nabla(\nabla\cdot\mathbf{V})$ was considered above. $(\nabla\cdot\nabla)\mathbf{V}$ is the Laplacian of $\mathbf{V}$ and is found to be

$$(\nabla\cdot\nabla)\mathbf{V} = \nabla^2\mathbf{V} = \mathbf{i}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial y^2} + \frac{\partial^2 V_x}{\partial z^2}\right) + \mathbf{j}\left(\frac{\partial^2 V_y}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_y}{\partial z^2}\right)$$

$$+ \mathbf{k}\left(\frac{\partial^2 V_z}{\partial x^2} + \frac{\partial^2 V_z}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2}\right).$$

12.10 Green's Theorem. Green's theorem supplies us with a formula that is sometimes very useful in solving some problems in vector algebra. The formula can be obtained by substituting $\mathbf{W} = U\nabla V$ in the divergence theorem.

$$\int \nabla\cdot\mathbf{W}\, dv = \int \mathbf{W}\cdot d\mathbf{a}.$$

The suggested substitution requires us to evaluate $\nabla\cdot(U\nabla V)$. To do this we recall the rule to differentiate a product: $(xy)' = xy' + x'y$ and we have

$$\nabla\cdot(U\nabla V) = (\nabla U)\cdot(\nabla V) + U\nabla^2 V.$$

The divergence theorem becomes

$$\int (\nabla U)\cdot(\nabla V)\, dv + \int U\nabla^2 V\, dv = \int U(\nabla V)\cdot d\mathbf{a}.$$

If we interchange U and V above we have

$$\int (\nabla V)\cdot(\nabla U)\,dv + \int V\nabla^2 U\,dv = \int V(\nabla U)\cdot d\mathbf{a}.$$

Now the difference of these expressions is known as Green's theorem and is

$$\int (U\nabla^2 V - V\nabla^2 U)\,dv = \int (U\nabla V - V\nabla U)\cdot d\mathbf{a}.$$

PROBLEMS ON CHAPTER 12

1. Prove the formulas

$$\nabla(UV) = U\nabla V + V\nabla U.$$
$$\nabla(\mathbf{U}\cdot\mathbf{V}) = \mathbf{U}\cdot\nabla\mathbf{V} + \mathbf{V}\cdot\nabla\mathbf{U} + \mathbf{U}\times(\nabla\times\mathbf{V}) + \mathbf{V}\times(\nabla\times\mathbf{U}).$$

2. Prove the formulas

$$\nabla\cdot(U\mathbf{V}) = U\nabla\cdot\mathbf{V} + \mathbf{V}\cdot\nabla U.$$
$$\nabla\cdot(\mathbf{U}\times\mathbf{V}) = \mathbf{V}\cdot\nabla\times\mathbf{U} - \mathbf{U}\cdot\nabla\times\mathbf{V}.$$

3. Prove the formulas

$$\nabla\times(U\mathbf{V}) = (\nabla U)\times\mathbf{V} + U(\nabla\times\mathbf{V}).$$
$$\nabla\times(\mathbf{U}\times\mathbf{V}) = \mathbf{V}\cdot\nabla\mathbf{U} - \mathbf{U}\cdot\nabla\mathbf{V} + \mathbf{U}\nabla\cdot\mathbf{V} - \mathbf{V}\nabla\cdot\mathbf{U}.$$

4. If U is a scalar field at each point equal to r^n, where r is the radius vector of the point, find ∇U.

5. If $\mathbf{U}$ is a vector field at each point equal to $r^n\mathbf{r}_1$, where $\mathbf{r}$ is the radius vector of the point, find $\nabla\cdot\mathbf{U}$ and $\nabla\times\mathbf{U}$.

6. If $\nabla\cdot\mathbf{B} = 0$ everywhere, prove

$$\nabla\times\{\nabla\times[\nabla\times(\nabla\times\mathbf{B})]\} = \nabla^2\nabla^2\mathbf{B}.$$

7. Prove that the volume enclosed by the surface A is equal to

$$\tfrac{1}{3}\int \mathbf{r}\cdot d\mathbf{a}$$

integrated over the surface A.

8. If A is a closed surface prove that

$$\int (\nabla\times\mathbf{U})\cdot d\mathbf{a} = 0, \qquad \int (\nabla U)\times d\mathbf{a} = 0.$$

9. Using the divergence theorem prove

$$\int (\nabla U)\times\mathbf{V}\cdot d\mathbf{a} = \int (\nabla U)\cdot\nabla\times\mathbf{V}\,dv.$$

Hint: Expand $\nabla\cdot(\nabla U\times\mathbf{V})$.

10. If $\mathbf{V}$ is the velocity of a point on a rigid body whose angular velocity is $\boldsymbol{\omega}$, show that

$$\nabla\times\mathbf{V} = 2\boldsymbol{\omega}.$$

CHAPTER 13

STRETCHED STRING AND ROUND DIAPHRAGM

13.1 Introduction. The study of natural vibrations in round members, such as telephone receiver diaphragms, loudspeaker cones, and round plates, is extremely important in the design of equipment suitable for use where there may be a tendency for such vibrations to occur. In a loudspeaker cone natural vibrations tend to amplify certain frequency ranges and therefore may be very objectionable. A drum head or bell produces a sound depending upon the frequencies of the natural vibrations and therefore the natural vibrations are all important.

If the vibrating member is round, the equation of motion turns out to be Bessel's equation and the solution is in Bessel's functions. The study of the vibration of a stretched string is a much simpler problem and just as important. If vibrations of guy wires become too great there is a danger that the wires may break. The method of solution of the two problems is the same. The stretched string has a solution in simple functions and will therefore be considered first. After the stretched string has been analyzed the same method will be used for the round diaphragm, and the presence of Bessel's functions will not complicate the discussion since the method has already been presented (Chapter 10).

The solution to the problem of the stretched string will give the displacement of a point on the string in terms of time and the location of the point along the string. There are, therefore, two independent variables, time and position, and we shall find we have partial derivatives in our equations.

13.2 Partial Derivatives. If u is a function of x, the ordinary derivative of u with respect to x is denoted by the symbol

$$\frac{du}{dx}$$

and is defined as

$$\lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

provided the limit exists. In the above expression $u(x + \Delta x)$ indicates that x has been replaced by $x + \Delta x$ in the expression for u. Therefore, $u(x)$ is not u times x which would be written xu or ux.

If u is a function of two variables x and y and if the following limit exists

$$\lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x}$$

it is defined as the partial derivative of u with respect to x and is denoted by the symbol

$$\frac{\partial u}{\partial x}$$

In obtaining the partial derivative of u with respect to x the value of y is treated as a constant.

Similarly, the partial derivative of u with respect to y is

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y}$$

provided the limit exists.

As illustrations, let $u = x^3 + 3x$.

$$\frac{du}{dx} = 3x^2 + 3.$$

This is the ordinary derivative since there is only one independent variable. Let $u = 2x^4 + 3x^2y + xy^3$.

$$\frac{\partial u}{\partial x} = 8x^3 + 6xy + y^3.$$

$$\frac{\partial u}{\partial y} = 3x^2 + 3xy^2.$$

Each of these partial derivatives is found by following the usual rules for ordinary differentiation; in the former case y is treated as a constant and in the latter case x is treated as constant.

If u is expressed as a function of two variables x and y, but y is known to be a function of x, then there is only one independent variable and to get the ordinary derivative of u with respect to x we need the following limit if it exists.

$$\frac{du}{dx} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) + u(x, y + \Delta y) - u(x, y)}{\Delta x}$$

$$\frac{du}{dx} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left[\frac{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)}{\Delta x} + \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} \frac{\Delta y}{\Delta x} \right]$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y)}{\Delta x} + \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}.$$

To illustrate the application of this formula, let $u = 2x^4 + 3x^2y + xy^3$ and let $y = 2x$. The formula gives for the derivative

$$\frac{du}{dx} = (8x^3 + 6xy + y^3) + (3x^2 + 3xy^2)(2).$$

When we substitute $y = 2x$ this becomes

$$\frac{du}{dx} = 8x^3 + 12x^2 + 8x^3 + 6x^2 + 24x^3 = 40x^3 + 18x^2.$$

To check this result substitute $y = 2x$ in the expression for u and then differentiate.

$$u = 2x^4 + 6x^3 + 8x^4 = 10x^4 + 6x^3.$$

$$\frac{du}{dx} = 40x^3 + 18x^2.$$

13.3 Total Differential. If u is a function of x, the differential of u is defined as

$$du = \frac{du}{dx} dx$$

where dx is an arbitrary increment in x and

$$\frac{du}{dx}$$

is the ordinary derivative. Note that the differential of u generally is not the same as an increment in u. Let $u = x^2$. Then

$$du = 2x\,dx$$

the increment in u in this case is

$$\begin{aligned}\Delta u &= (x + dx)^2 - x^2 \\ &= x^2 + 2x\,dx + (dx)^2 - x^2 \\ &= 2x\,dx + (dx)^2.\end{aligned}$$

The differential is an increment measured on the tangent line rather than on the curve itself.

If u is a function of two variables, the graph of u plotted against x and y may be considered as a surface. The total differential in u is an increment in u measured on the tangent plane when x and y are subject to increments, that is

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

13.4 Partial Differential Equations. If we differentiate

$$u = x^3 + 3x^2 - 4$$

we get

$$\frac{du}{dx} = 3x^2 + 6x.$$

This is an ordinary differential equation the general solution of which is

$$u = x^3 + 3x^2 + C.$$

In order to show that the integration constant is -4 we require a boundary condition.

Let us see what the corresponding example is with more than one independent variable. Let us differentiate

$$u = x^3 + 3x^2y^3 + 4y^2 + 3y + 2$$

with respect to x. This gives us

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy^3.$$

If we attempt to integrate this partial differential equation, we have

$$u = x^3 + 3x^2y^3 + f(y)$$

and we have to depend on boundary conditions to show that the function of integration $f(y) = 4y^2 + 3y + 2$.

As another example, let

$$u = x^2 + 2xy + y^2.$$

$$\frac{\partial u}{\partial x} = 2x + 2y.$$

$$\frac{\partial u}{\partial y} = 2x + 2y.$$

Therefore, u satisfies the equation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}.$$

Now let

$$u = (x + y)^3 + \sin (x + y).$$

$$\frac{\partial u}{\partial x} = 3(x + y)^2 + \cos (x + y).$$

$$\frac{\partial u}{\partial y} = 3(x + y)^2 + \cos (x + y).$$

This value of u also satisfies the same differential equation

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}.$$

If we were given this partial differential equation and were required to find an expression for u to satisfy the equation, we would have to depend on the boundary conditions almost entirely. Any function of $x + y$ will satisfy the differential equation above.

In this chapter and the following chapter four examples involving partial differential equations are worked out. Attention is called to the effect of applying the boundary conditions, for example, that the ends of the string are stationary and that the edge of the round diaphragm is stationary. The motions obtained for the string and diaphragm are free motions. The equation for the driven diaphragm might be quite different from that obtained. In the cases of skin effect only the effect with alternating current is considered. The skin effect produced by a switching transient or lightning surge passing along a wire would be a different matter.

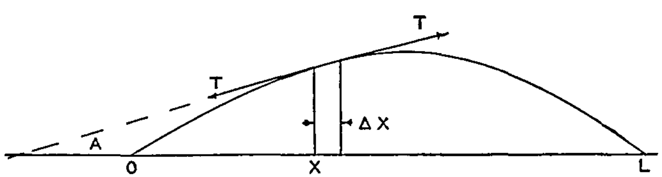

FIG. 13-1

13.5 Equation of Motion of Stretched String. Consider a string of uniform construction so that the mass per unit length ρ is the same over the entire length. The string is fastened at the ends and is under tension T. The length of the string is L. Assume that when the string vibrates, the vibrations are so small that the tension remains constant. We shall neglect the effect of air on the string and neglect all friction effects.

Let x be measured along the equilibrium position of the string with origin at one end of the string; let y be the deflection of the string at any place x at time t. Note that y is a function of x and t, and if any differentiation is performed we must use partial derivatives. Consider an element of the string of length Δx. See Fig. 13-1. The direction of the tension on the left of Δx is along the tangent to the curve at x. Let the angle between the tangent and the x axis be A. Since the deflection of the string is small A will be small, and therefore $\tan A$, A, and $\sin A$ will all be practically equal. Therefore

$$\sin A = \tan A = \frac{\partial y}{\partial x}. \tag{13.1}$$

The downward component of the tension on the left of Δx is

$$T \sin A = T \frac{\partial y}{\partial x}. \tag{13.2}$$

The upward component of the tension on the right of Δx may be different from the above since the slope of the curve may be different. It can be found by adding to the term in equation (13.2) an extra term to take into account the change produced in $\sin A$ when it is measured a distance Δx to the right. The result is

$$T \sin A + \frac{\partial}{\partial x}(T \sin A)\Delta x = T\frac{\partial y}{\partial x} + T\frac{\partial^2 y}{\partial x^2}\Delta x. \tag{13.3}$$

The resultant upward force will be found by subtracting the force in equation (13.2) from the force in equation (13.3) and is

$$F = T\frac{\partial^2 y}{\partial x^2}\Delta x. \tag{13.4}$$

Equating this to the mass times the acceleration of the element we have

$$T\frac{\partial^2 y}{\partial x^2}\Delta x = \rho\Delta x\frac{\partial^2 y}{\partial t^2}. \tag{13.5}$$

If equation (13.5) is divided by Δx and the element Δx allowed to approach zero, we obtain the general equation of the motion of a stretched string.

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}. \tag{13.6}$$

13.6 Harmonic Vibrations. Let us make the following assumption: the manner in which the motion depends on time is a sine function. We assume, therefore, that

$$y = Y \sin(\omega t + B) \tag{13.7}$$

where Y takes care of the variation with respect to x. ω and B are constants whose values are not yet known. We substitute equation (13.7) into equation (13.6) and try to obtain Y as a function of x. To do this we require derivatives of y and they are listed below for the convenience of the student.

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{dY}{dx}\sin(\omega t + B). \\
\frac{\partial^2 y}{\partial x^2} &= \frac{d^2Y}{dx^2}\sin(\omega t + B). \\
\frac{\partial y}{\partial t} &= Y\omega\cos(\omega t + B). \\
\frac{\partial^2 y}{\partial t^2} &= -Y\omega^2\sin(\omega t + B).
\end{aligned} \tag{13.8}$$

Substituting in the equation of motion (13.6), we have

$$T\frac{d^2Y}{dx^2}\sin(\omega t + B) = -\rho\omega^2 Y\sin(\omega t + B). \tag{13.9}$$

Dividing by $T\sin(\omega t + B)$, we have

$$\frac{d^2Y}{dx^2} + \frac{\rho\omega^2}{T}Y = 0. \tag{13.10}$$

This is a homogeneous linear differential equation. Any expression for Y that will satisfy equation (13.10) will make y in equation (13.7) satisfy the equation of motion (13.6). The characteristic equation for (13.10) is

$$m^2 + \frac{\rho\omega^2}{T} = 0. \tag{13.11}$$

Therefore, we have for m

$$m = \pm i\omega\sqrt{\frac{\rho}{T}}. \tag{13.12}$$

We can now write a solution for equation (13.10)

$$Y = C_1\sin\omega x\sqrt{\frac{\rho}{T}} + C_2\cos\omega x\sqrt{\frac{\rho}{T}}. \tag{13.13}$$

Since the string is fastened at both ends we know that $Y = 0$ when $x = 0$ and when $x = L$. To satisfy the first condition, C_2 must be zero. Substituting the second condition into equation (13.13) gives

$$0 = C_1\sin\omega L\sqrt{\frac{\rho}{T}}. \tag{13.14}$$

For this to be satisfied, either $C_1 = 0$ and there is no vibration, or

$$\omega L \sqrt{\frac{\rho}{T}} = N\pi \tag{13.15}$$

where N is any integer. From this we have

$$\omega = \frac{N\pi}{L\sqrt{\frac{\rho}{T}}}. \tag{13.16}$$

Equation (13.13) can now be written

$$Y = C_1 \sin \frac{N\pi x}{L}. \tag{13.17}$$

In Fig. 13–2 we have plotted the curve of equation (13.17) for the three cases $N = 1$, $N = 2$, $N = 3$.

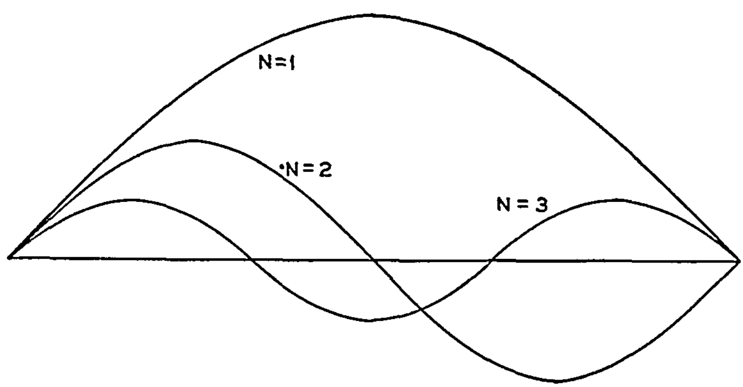

FIG. 13–2

Substituting equations (13.16) and (13.17) into (13.7), we have for a solution

$$y = C_1 \sin \frac{N\pi x}{L} \sin \left[\frac{N\pi t}{L\sqrt{\frac{\rho}{T}}} + B \right]. \tag{13.18}$$

ω in equation (13.16) is $2\pi f$. Therefore

$$f = \frac{N}{2L}\sqrt{\frac{T}{\rho}}. \tag{13.19}$$

This equation shows that the frequency is increased by decreasing the length of the string, increasing the tension and by using lighter material

or a string of smaller cross section. The value of N in equation (13.19) depends upon the type of vibration. If the string is vibrating as shown in the curve marked $N = 1$ in Fig. 13–2, $N = 1$ in equation (13.19). If the mode of vibration is as shown by the curve marked $N = 2$ in Fig. 13–2, $N = 2$ in equation (13.19) and the frequency will be double that for $N = 1$, etc. If $N = 1$ the string vibrates at the lowest possible frequency and this is called the fundamental frequency. If $N = 2$ the vibration is the second harmonic, and for $N = 3$ the vibration is the third harmonic, etc. In the case of the stretched string the frequency of the Nth harmonic is N times the frequency of the fundamental.

13.7 Fourier Series Solution. Since equation (13.18) gives a different solution of the equation of motion (13.6), for each integer value of N, we have an infinite number of solutions. Since equation (13.6) is a linear equation and the sum of two or more solutions is also a solution we can now write for a solution

$$y = C_1 \sin \frac{\pi x}{L} \sin \left(\frac{\pi t}{L} \sqrt{\frac{T}{\rho}} + B_1 \right) + C_2 \sin \frac{2\pi x}{L} \sin \left(\frac{2\pi t}{L} \sqrt{\frac{T}{\rho}} + B_2 \right) + \cdots. \tag{13.20}$$

The velocity of the string at any point is

$$\frac{\partial y}{\partial t} = \frac{C_1 \pi}{L} \sqrt{\frac{T}{\rho}} \sin \frac{\pi x}{L} \cos \left(\frac{\pi t}{L} \sqrt{\frac{T}{\rho}} + B_1 \right) + \frac{2C_2 \pi}{L} \sqrt{\frac{T}{\rho}} \sin \frac{2\pi x}{L} \cos \left(\frac{2\pi t}{L} \sqrt{\frac{T}{\rho}} + B_2 \right) + \cdots. \tag{13.21}$$

If we know the configuration of the string and the velocity of every point of the string at any time, we can determine all the B's and C's in equations (13.20) and (13.21) and the problem is completely solved. Some numerical examples follow.

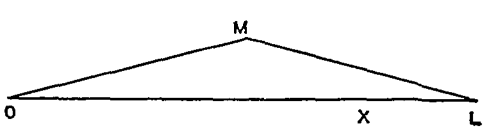

FIG. 13–3

13.8 Examples of Stretched String. Suppose the string is displaced at its middle a distance M as shown in Fig. 13–3, and at $t = 0$ the middle is released and the string is allowed to vibrate freely.

Since the string is stationary at $t = 0$, all the B's equal $\pi/2$, or we can write equation (13.20) using cosines instead of sines.

$$y = \sum_{N=1}^{\infty} C_N \sin \frac{N\pi x}{L} \cos \frac{N\pi t}{L} \sqrt{\frac{T}{\rho}}. \tag{13.22}$$

At $t = 0$, equation (13.22) is

$$y_0 = \sum_{N=1}^{\infty} C_N \sin \frac{N\pi x}{L}. \tag{13.23}$$

Now equation (13.23) is the Fourier series for the function of Fig. 13.3. Since equation (13.23) is a sine series, an odd function, the curve to be analyzed should be an odd function. Figure 13–4 shows how the function

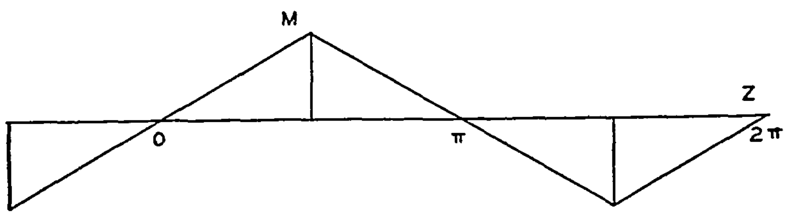

FIG. 13–4

in Fig. 13–3 should be extended to make it a periodic odd function. If we define a new variable z so that

$$z = \frac{\pi x}{L} \tag{13.24}$$

equation (13.23) becomes

$$y_0 = \sum_{N=1}^{\infty} C_N \sin Nz \tag{13.25}$$

and we can find the C's from the formula

$$C_N = \frac{1}{\pi} \int_0^{2\pi} f(z) \sin Nz \, dz. \tag{13.26}$$

Substituting for $f(z)$, we have

$$C_N = \frac{1}{\pi} \int_0^{\pi/2} \frac{2Mz}{\pi} \sin Nz \, dz + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} 2M \left(1 - \frac{z}{\pi}\right) \sin Nz \, dz$$
$$+ \frac{1}{\pi} \int_{3\pi/2}^{2\pi} 4M \left(\frac{z}{2\pi} - 1\right) \sin Nz \, dz. \tag{13.27}$$

The result of performing the integration in equation (13.27) shows that, if N is even, $C_N = 0$ and, if N is odd, we have

$$C_N = \frac{8M}{\pi^2 N^2} \quad \text{for } N = 1, 5, 9, \cdots. \tag{13.28}$$

$$C_N \doteq -\frac{8M}{\pi^2 N^2} \quad \text{for } N = 3, 7, 11, \cdots. \tag{13.29}$$

These can be combined into

$$C_N = (-1)^{\frac{N-1}{2}} \frac{8M}{\pi^2 N^2} \tag{13.30}$$

where N is odd.

The final result is obtained by substituting in equation (13.22).

$$y = \frac{8M}{\pi^2} \sum_{N=1}^{\infty} \frac{(-1)^{\frac{N-1}{2}}}{N^2} \sin \frac{N\pi x}{L} \cos \frac{N\pi t}{L}\sqrt{\frac{T}{\rho}} \tag{13.31}$$

where N is odd.

If the string is set in vibration in this manner the amplitude of the third harmonic is only 11 per cent that of the fundamental, the fifth harmonic is 4 per cent, the seventh harmonic 2 per cent, etc. If the string were held at a place other than the middle, the form of the function Y would be different from that in Fig. 13-3 and therefore the Fourier series should be different and the harmonics would have different percentages. In the case just worked out, where the string was held in the middle, there are no even harmonics. If the string had been plucked one-third the distance from one end there would be no harmonics that are multiples of three. If the string is plucked, bowed, or struck with a mallet, the position along the string at which it is actuated affects the relative magnitudes of the harmonics. This accounts for the production of "sour notes" on the violin by the tyro.

13.9 Equation of Motion of a Round Diaphragm. The analysis of this problem is based upon the same type of assumptions as made for the preceding problem. The diaphragm is uniform everywhere, its density is ρ mass per unit volume, its thickness is h, and it is stretched to a tension of T per unit area. The deflection at any point is y and is everywhere so small that T, ρ, and h remain constant. The diaphragm has a radius R and is clamped so that its periphery is stationary. Let x be the distance measured from the center of the diaphragm, and consider a circular element between x and $x + \Delta x$. See Fig. 13-5.

We shall restrict our study to the case where all particles in the circular element are moving together. The upward component of the tension inside the element is

$$F_1 = 2\pi x h T \sin B \tag{13.32}$$

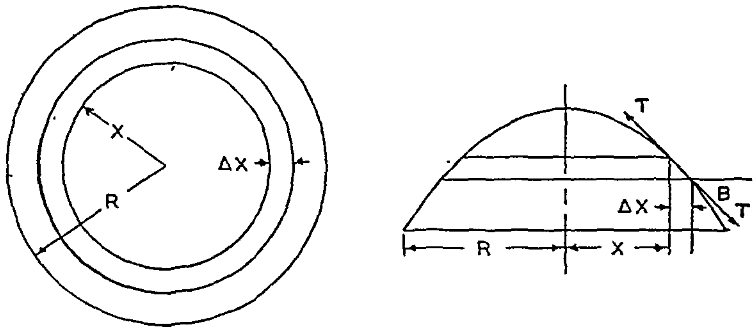

FIG. 13-5

where B is the angle between the tangent to the curve and the x axis. Since the deflection is small B is small and

$$\sin B = B = \tan B = -\frac{\partial y}{\partial x}. \tag{13.33}$$

Therefore, we can write the upward component

$$F_1 = -2\pi x h T \frac{\partial y}{\partial x}. \tag{13.34}$$

To determine the downward component of the force on the outer edge of the element note that both x and the partial derivative in equation (13.34) can change if we move to the outer edge of the element. The downward component of the force on the outer edge is found from

$$F_2 = F_1 + \frac{\partial F_1}{\partial x}\Delta x \tag{13.35}$$

$$= -2\pi h x T \frac{\partial y}{\partial x} + \frac{\partial}{\partial x}\left(-2\pi x h T \frac{\partial y}{\partial x}\right)\Delta x \tag{13.36}$$

$$= -2\pi h x T \frac{\partial y}{\partial x} - \left(2\pi h T \frac{\partial y}{\partial x} + 2\pi h T x \frac{\partial^2 y}{\partial x^2}\right)\Delta x. \tag{13.37}$$

The resultant upward force is

$$F_1 - F_2 = 2\pi h T\left(\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right)\Delta x. \tag{13.38}$$

The mass of the element is $2\pi\rho xh\Delta x$. Equating mass times acceleration to the resultant force, we have

$$2\pi h\rho x \frac{\partial^2 y}{\partial t^2}\Delta x = 2\pi hT\left(\frac{\partial y}{\partial x} + x\frac{\partial^2 y}{\partial x^2}\right)\Delta x. \tag{13.39}$$

If this is divided by $2\pi h\Delta x$, and Δx is allowed to approach zero, we have the equation of motion of the round diaphragm.

$$\rho x \frac{\partial^2 y}{\partial t^2} = T\frac{\partial y}{\partial x} + Tx\frac{\partial^2 y}{\partial x^2}. \tag{13.40}$$

It is customary to multiply this equation by x and divide it by T to obtain a special form. The result, when rearranged, is

$$x^2\frac{\partial^2 y}{\partial x^2} + x\frac{\partial y}{\partial x} - \frac{\rho}{T}x^2\frac{\partial^2 y}{\partial t^2} = 0. \tag{13.41}$$

13.10 First Solution for Round Diaphragm. As in the case of the string we assume that the motion varies in time as a sine function.

$$y = Y\sin(\omega t + \phi) \tag{13.42}$$

where Y takes care of the variation in space, and therefore Y is a function of x but not of t. The derivatives of y can be written in terms of Y and t as follows:

$$\begin{aligned}
\frac{\partial y}{\partial x} &= \frac{dY}{dx}\sin(\omega t + \phi)\\
\frac{\partial^2 y}{\partial x^2} &= \frac{d^2Y}{dx^2}\sin(\omega t + \phi)\\
\frac{\partial y}{\partial t} &= Y\omega\cos(\omega t + \phi)\\
\frac{\partial^2 y}{\partial t^2} &= -Y\omega^2\sin(\omega t + \phi)
\end{aligned} \tag{13.43}$$

If these are substituted into equation (13.42), and the factor $\sin(\omega t + \phi)$ which appears in every term divided out, we have

$$x^2\frac{d^2Y}{dx^2} + x\frac{dY}{dx} + \frac{\rho\omega^2}{T}x^2Y = 0. \tag{13.44}$$

We recognize this as Bessel's equation of order zero. If we let $a^2 = \omega^2\rho/T$, the solution is

$$Y = C_1J_0(ax) + C_2N_0(ax) \tag{13.45}$$

where C_1 and C_2 are arbitrary constants that depend on the boundary conditions.

It happens that $N_0(ax)$ is infinite for $x = 0$. Therefore $C_2 = 0$ since we have agreed to consider small free vibrations. Now, where $x = R$, $Y = 0$ corresponding to the stationary rim of the diaphragm. Therefore

$$0 = C_1 J_0(aR). \tag{13.46}$$

Table X–3 gives the first 20 values of aR that will satisfy equation (13.46), for example,

$$aR = 2.40,\ 5.52,\ 8.65, \text{ etc.}$$

Substituting for a the value assigned it above, we have

$$\omega R \sqrt{\frac{\rho}{T}} = 2.40,\ 5.52, \text{ or } 8.65, \text{ etc.} \tag{13.47}$$

Now R, ρ, and T are known; therefore equation (13.47) enables us to determine ω. The values of ω so determined will tell what the natural frequencies are. Our solution can be written

$$y = C_1 J_0\left(\omega x \sqrt{\frac{\rho}{T}}\right) \sin(\omega t + \phi). \tag{13.48}$$

13.11 Harmonic Frequencies. The frequency of vibration f of the diaphragm is given by $2\pi f = \omega$ and ω is determined from equation (13.47). The lowest frequency possible corresponds to the root 2.40 and is

$$f_1 = \frac{1}{2\pi R}\sqrt{\frac{T}{\rho}}\, 2.40.$$

The next lowest frequency is

$$f_2 = \frac{1}{2\pi R}\sqrt{\frac{T}{\rho}}\, 5.52.$$

The ratio of the second lowest to the lowest frequency is $5.52/2.40 = 2.3$. This is quite different from the case of the stretched string where we found the ratio of the frequencies of the second harmonic to the fundamental to be exactly 2.

13.12 More Complete Solution for Round Diaphragm. Each value of ω found by means of equation (13.47) will make equation (13.48) a solution to the differential equation (13.41). Since there is an infinite number of roots to the equation

$$J_0(ax) = 0$$

there will be an infinite number of values of ω given by equation (13.47)

and therefore an infinite number of solutions to the differential equation. Since the differential equation is a linear equation, the sum of two or more solutions is also a solution, and therefore we can write as a more general solution

$$y = C_1 J_0\left(\omega_1 x \sqrt{\frac{\rho}{T}}\right) \sin(\omega_1 t + \phi_1) + C_2 J_0\left(\omega_2 x \sqrt{\frac{\rho}{T}}\right) \sin(\omega_2 t + \phi_2)$$
$$+ C_3 J_0\left(\omega_3 x \sqrt{\frac{\rho}{T}}\right) \sin(\omega_3 t + \phi_3) + \cdots. \quad (13.49)$$

The velocity at any radius x is obtained by taking the partial derivative of equation (13.49) with respect to t.

$$\frac{\partial y}{\partial t} = C_1\omega_1 J_0\left(\omega_1 x \sqrt{\frac{\rho}{T}}\right) \cos(\omega_1 t + \phi_1) + C_2\omega_2 J_0\left(\omega_2 x \sqrt{\frac{\rho}{T}}\right) \cos(\omega_2 t + \phi_2)$$
$$+ C_3\omega_3 J_0\left(\omega_3 x \sqrt{\frac{\rho}{T}}\right) \cos(\omega_3 t + \phi_3) + \cdots. \quad (13.50)$$

If we choose some point of the diaphragm x, the series in equations (13.49) and (13.50) are Fourier series for the deflection and velocity of that point as functions of time. If we choose any instant of time, the series are series of Bessel's functions of order zero of the first kind. If the deflection of each point and the velocity of each point are known for any one instant of time, the C's and the ω's can be computed using the method of section 10.17 in Chapter 10.

PROBLEMS ON CHAPTER 13

1. A steel wire 0.005 in. in diameter (specific gravity = 7.83, or density = 489 lbs. per cu. ft.) is stretched between two edges 3 ft. apart with a force of 15 lbs. What will be the lowest natural frequency?

2. If the wire in problem one is to be stretched so that the fundamental frequency is 500 cycles per second, what force will be required?

3. If the safe working stress of the steel in problem 1 is 40,000 lbs. per sq. in., how high can the fundamental frequency be made?

4. Perform the integration indicated in equation (13.27) and check equation (13.30).

5. Work the example in section 13.8 if the string is displaced a distance M at a point one-third the distance from one end, and at $t = 0$ this point is released.

6. What is the ratio of the amplitude of the fourth harmonic to the amplitude of the fundamental in problem 5?

7. What point of the string should be given the maximum displacement (see the example in section 13.8) so that the amplitude of the second harmonic will be equal to one-fourth the amplitude of the fundamental?

8. Solve problem 7 where the amplitude of the third harmonic is to be made equal to the amplitude of the second harmonic.

9. Work the example of section 13.8 where the string is not displaced but at $t = 0$ every point of the middle third is given a velocity v.

10. A round diaphragm is made of sheet steel 0.05 in. thick and 4 in. in diameter. If it is stretched to a tension of 30,000 lb. per sq. in., what will be the frequency of the fundamental?

11. The diaphragm in problem 10 is replaced by a brass diaphragm of the same size. If brass has a density of 534 lbs. per cu. ft., what tension should be applied to give the brass the same frequency?

12. If a diaphragm 3 in. in diameter and 0.005 in. thick is to be stretched so that its fundamental frequency is as high as possible, which material listed below should be used?

Material	Pounds per Cubic Foot	Maximum Stress pounds per square inch
Steel	489	40,000
Brass	534	45,000
Bronze	554	40,000
Nickel	537	65,000

13. Rate the materials in problem 12 on the basis of the value of the fundamental frequency. Give the ratio of the fundamental frequency for each material to the fundamental frequency of the material having the highest fundamental.

14. If the frequency of the second harmonic of one string is twice the frequency of the fundamental of another string, what will be the ratio of the fundamental frequencies?

15. If the frequency of the second harmonic of one round diaphragm is twice the frequency of the fundamental of another round diaphragm, what will be the ratio of the fundamentals?

16. If the frequency of the second harmonic of a round diaphragm is double the frequency of the fundamental of a string, what will be the ratio of the fundamentals?

17. If the frequency of the third harmonic of one round diaphragm is three-fifths of the frequency of the fifth harmonic of another round diaphragm, what will be the ratio of the fundamentals?

18. If the frequency of the second harmonic of one round diaphragm is equal to the fundamental of another what will be the ratio of the third harmonics?

19. What will be the ratio of the fourth harmonics in problem 18?

20. In problem 18 what will be the ratio of the frequency of the fundamental of the first to the second harmonic of the second diaphragm?

21. If the fundamental frequency of a round diaphragm is equal to the fundamental frequency of a string, what will be the ratio of like harmonics, e.g., third, fourth, etc.?

22. If the fundamental frequency of one string is twice the fundamental frequency of another string, what will be the ratio of the second harmonics? Third? Fourth?

23. If the fundamental frequency of a round diaphragm is twice the fundamental frequency of a string, what will be the ratio of the second harmonics? Third? Fourth?

24. If the fundamental frequency of one round diaphragm is twice the fundamental frequency of another round diaphragm, what will be the ratio of the second harmonics? Third? Fourth?

CHAPTER 14

SKIN EFFECT PROBLEMS

14.1 Introduction. If the current in a cylindrical conductor is constant or is varying at a low rate, the current density will be practically uniform over the cross section. However, if the current is varying at a high rate, the current density will no longer be uniform but will be greater near the outer surface of the conductor, producing what is called skin effect. This crowding of the current in the skin or outer surface of the conductor increases the effective resistance of the conductor so that resistance measurements made with direct current or at low frequencies are of little value if the conductor is used at high frequency.

14.2 Flat Conductor. High current circuits frequently are made up of flat conductors to take advantage of their mechanical flexibility and improved space factor in the slots of electrical machinery. Figure 14–1 shows the cross section of a flat conductor of width A and thickness B. The direction of current is perpendicular to the plane of the page. The flux set up outside the conductor is produced by all the current in the conductor, and any change in this flux induces a voltage which is the same throughout the cross section. The flux inside the conductor is produced by only part of the current and, therefore, changes in the internal flux induce voltages over only part of the cross section. If A is large compared to B the flux path (see Fig. 14–1) may be assumed to be of length $2A$ and we may neglect the influence of the shape of the path at the ends of the cross section.

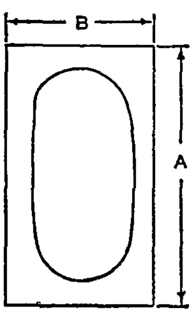

FIG. 14–1

14.3 Equation for Flat Conductor. The mathematical treatment which follows shows how the skin effect in flat conductors can be determined on the basis of the following restrictions.

1. Steady conditions have been established.
2. Currents and voltages are sine functions of time.
3. The return conductor is far enough away so that its effect is negligible.
4. The permeability of the conductor and of the space around the conductor is constant and uniform.

Referring to Fig. 14–1, we set up a reference axis Ox perpendicular to the wide surface and with the origin halfway between sides. Let D be the

current density at a distance x from the center line of the conductor. The total current in the area within a distance x from the center line is

$$i_x = 2A \int_0^x D\,dx. \tag{14.1}$$

The length of the flux path is $2A$, neglecting end effect. Therefore

$$H_x = \frac{i_x}{2A} = \int_0^x D\,dx. \tag{14.2}$$

If μ is the permeability of the conductor, the flux density at a distance x from the center line is

$$B_x = \mu H_x = \mu \int_0^x D\,dx. \tag{14.3}$$

We can eliminate the integral sign by differentiating with respect to x

$$\frac{\partial B_x}{\partial x} = \mu D. \tag{14.4}$$

The partial derivative is used since B_x is a function of both position and time.

Figure 14–2 shows the edge of the conductor of length S. The flux in the elementary area $EFGH$ is

$$\Delta\phi = B_x S \Delta x. \tag{14.5}$$

The voltage induced around the circuit $EFGHE$, when $\Delta\phi$ changes, is

$$v = \frac{d(\Delta\phi)}{dt} = S\frac{\partial B_x}{\partial t}\Delta x. \tag{14.6}$$

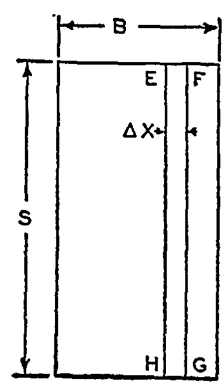

FIG. 14–2

The resistance drop along EH is ρSD where ρ is the resistivity of the conductor material. The resistance drop along FG, which is a distance Δx farther from the center line of the conductor, is

$$\rho S\left(D + \frac{\partial D}{\partial x}\Delta x\right).$$

The resistance drops along EF and GH are zero since the current is along the length of the conductor. The resistance drop around $EFGHE$ is

$$\rho S\left(D + \frac{\partial D}{\partial x}\Delta x\right) - \rho SD = \rho S\frac{\partial D}{\partial x}\Delta x. \tag{14.7}$$

The resistance drop must be equal to the voltage induced or

$$\rho S \frac{\partial D}{\partial x} \Delta x = S \frac{\partial B_x}{\partial t} \Delta x. \tag{14.8}$$

$$\rho \frac{\partial D}{\partial x} = \frac{\partial B_x}{\partial t}. \tag{14.9}$$

Differentiate equation (14.9) with respect to x.

$$\rho \frac{\partial^2 D}{\partial x^2} = \frac{\partial^2 B_x}{\partial x \partial t}. \tag{14.10}$$

Differentiate equation (14.4) with respect to t

$$\frac{\partial^2 B_x}{\partial t \partial x} = \mu \frac{\partial D}{\partial t}. \tag{14.11}$$

The last two equations give

$$\rho \frac{\partial^2 D}{\partial x^2} = \mu \frac{\partial D}{\partial t}. \tag{14.12}$$

This is the differential equation for the general problem of a flat conductor.

14.4 Special Case for Alternating Current. Our problem now is to find an expression for D as a function of x and a sine function in time that will satisfy equation (14.12). We prefer to use exponential expressions rather than sine functions in this problem since it is easier. We shall first establish the following:

If $D' + iD$ satisfies equation (14.12), where D' and D are real, D' and D will each satisfy the same equation. If $D' + iD$ satisfies equation (14.12), we have

$$\rho \frac{\partial^2 (D' + iD)}{\partial x^2} - \mu \frac{\partial (D' + iD)}{\partial t} = 0. \tag{14.13}$$

This can be arranged as follows:

$$\rho \frac{\partial^2 D'}{\partial x^2} + i\rho \frac{\partial^2 D}{\partial x^2} - \mu \frac{\partial D'}{\partial t} - i\mu \frac{\partial D}{\partial t} = 0. \tag{14.14}$$

$$\left[\rho \frac{\partial^2 D'}{\partial x^2} - \mu \frac{\partial D'}{\partial t}\right] + i \left[\rho \frac{\partial^2 D}{\partial x^2} - \mu \frac{\partial D}{\partial t}\right] = 0. \tag{14.15}$$

The real part on the left and the imaginary part on the left are each equal to zero, or

$$\rho \frac{\partial^2 D'}{\partial x^2} - \mu \frac{\partial D'}{\partial t} = 0, \tag{14.16}$$

$$\rho \frac{\partial^2 D}{\partial x^2} - \mu \frac{\partial D}{\partial t} = 0 \tag{14.17}$$

showing that, if we can find a complex expression that satisfies equation (14.12), the real part and the imaginary part will satisfy the same equation (14.12). Let us try

$$D' + iD = y\, e^{i\omega t}$$

in equation (14.12) and see if a value of y in terms of x can be found to make it satisfy the equation.

$$\rho \frac{d^2y}{dx^2} e^{i\omega t} - i\mu\omega y\, e^{i\omega t} = 0. \tag{14.18}$$

The partial derivative with respect to x in equation (14.12) becomes the ordinary derivative in equation (14.18) since y is a function of x alone. The partial derivative with respect to t does not appear in equation (14.18) since we can differentiate $y\, e^{i\omega t}$ with respect to t and obtain $i\omega y\, e^{i\omega t}$. Equation (14.18) becomes

$$\frac{d^2y}{dx^2} - i\frac{\mu\omega}{\rho} y = 0. \tag{14.19}$$

If we can find a value for y in terms of x to satisfy equation (14.19), $y\, e^{i\omega t}$ will satisfy equation (14.12) and the imaginary part of $y\, e^{i\omega t}$ will also satisfy (14.12). Suppose that we find $y = y_1 + iy_2$ satisfies equation (14.19). Then $ye^{i\omega t} = (y_1 + iy_2)(\cos\omega t + i\sin\omega t) = (y_1\cos\omega t - y_2\sin\omega t) + i(y_1\sin\omega t + y_2\cos\omega t)$ satisfies equation (14.12) and our solution at this step would be $y_1\sin\omega t + y_2\cos\omega t$.

Let

$$\frac{\mu\omega}{2\rho} = m^2. \tag{14.20}$$

Equation (14.19) then becomes

$$\frac{d^2y}{dx^2} - i2m^2y = 0. \tag{14.21}$$

This is a homogeneous linear differential equation with constant coefficients. The general solution is

$$y = K_1e^{m\sqrt{2i}\,x} + K_2e^{-m\sqrt{2i}\,x}. \tag{14.22}$$

The symmetry of the problem tells us that the effect is the same on both sides of $x = 0$, or $K_1 = K_2 = K$, and

$$y = K(e^{m\sqrt{2i}\,x} + e^{-m\sqrt{2i}\,x}). \tag{14.23}$$

$$y = 2K\cosh m\sqrt{2i}\,x. \tag{14.24}$$

14.5 Three Special Cases. There are three special cases depending upon the value of m in equation (14.23). These cases are illustrated in

Fig. 14–3 which shows the maximum current density for each level in the conductor.

If m is small we have practically the case of direct current where the current density is uniform throughout the cross section of the conductor.

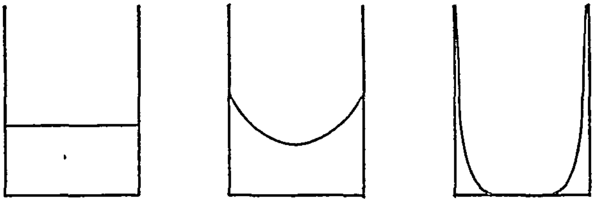

FIG. 14–3

If m is large enough so that $e^{0.5mB}$ is large compared to unity, $e^{-m\sqrt{2i}\,x}$ can be neglected in equation (14.23) and we have in this case

$$
\begin{aligned}
y &= Ke^{m\sqrt{2i}\,x} \qquad x > 0, \\
y &= Ke^{-m\sqrt{2i}\,x} \qquad x < 0.
\end{aligned} \tag{14.25}
$$

The case for medium values of m does not simplify as much as the others and will not be considered at this time.

Taking the case of higher frequency where m is larger, we have

$$y = Ke^{m\sqrt{2i}\,x} = Ke^{m(1+i)x} = Ke^{mx}e^{imx} \qquad x > 0, \tag{14.26}$$

$$y = Ke^{mx}(\cos mx + i \sin mx). \tag{14.27}$$

Now D, the current density, is the imaginary part of $y\,e^{i\omega t}$, or

$$D = Ke^{mx}(\cos mx \sin \omega t + \sin mx \cos \omega t) \tag{14.28}$$

$$= Ke^{mx}\sin(mx + \omega t) \qquad x > 0. \tag{14.29}$$

We see that the current density reaches a maximum at different levels at different instants of time. If we integrate equation (14.29) from $x = 0$ to $x = 0.5B$, we will get half the total current in the conductor. However, since m is large, equation (14.29) gives negligible values for D when x is zero and for negative values of x. We therefore integrate equation (14.29) from $x = -\infty$ to $x = 0.5B$ instead of from $x = 0$ to $x = 0.5B$.

$$i = 2AK\int_{-\infty}^{0.5B} e^{mx}\sin(mx + \omega t)\,dx \tag{14.30}$$

$$= \frac{AK}{m}e^{0.5mB}[\sin(\omega t + 0.5mB) - \cos(\omega t + 0.5mB)] \tag{14.31}$$

$$= \frac{AK}{m}e^{0.5mB}\left[\sin(\omega t + 0.5mB) - \sin\left(\omega t + 0.5mB + \frac{\pi}{2}\right)\right]. \tag{14.32}$$

In vector form we have

$$\mathbf{I} = \frac{AK}{m\sqrt{2}} e^{0.5mB} (1 - i)e^{i0.5mB}. \tag{14.33}$$

The current density at the surface of the conductor is found by substituting $x = 0.5B$ in equation (14.29). This gives

$$D_0 = Ke^{0.5mB} \sin (\omega t + 0.5mB). \tag{14.34}$$

The voltage drop per unit length equals the resistivity times the current density at the surface, or

$$v = K\rho e^{0.5mB} \sin (\omega t + 0.5mB). \tag{14.35}$$

In vector form we have

$$\mathbf{V} = \frac{K\rho}{\sqrt{2}} e^{0.5mB} e^{i0.5mB}. \tag{14.36}$$

The impedance $\mathbf{Z}$ is equal to the voltage $\mathbf{V}$ divided by the current $\mathbf{I}$. Therefore

$$Z = R_{ac} + iX = \frac{\rho m}{A(1 - i)} = \frac{\rho m(1 + i)}{2A} = \frac{\rho m}{2A} + i\frac{\rho m}{2A}. \tag{14.37}$$

Therefore the resistance per unit length to alternating current, R_{ac}, is found to be

$$R_{ac} = \frac{\rho m}{2A}.$$

The resistance per unit length for direct current, R_{dc}, is given by

$$R_{dc} = \frac{\rho}{AB}.$$

The ratio of the resistances is

$$\frac{R_{ac}}{R_{dc}} = \frac{mB}{2} = \frac{B}{2}\sqrt{\frac{\mu\omega}{2\rho}} \tag{14.38}$$

for large values of m.

There is an interesting relation that becomes evident when we write equation (14.31) as follows:

$$i = \frac{AK\sqrt{2}}{m} e^{0.5mB} \sin (\omega t + 0.5mB - 45°). \tag{14.39}$$

If this is compared with equation (14.34), we note that the total current lags the current density at the surface of the conductor by 45°. It is

evident that measurements made by using test leads to touch parts of the circuit are unreliable.

14.6 Equation for Round Conductor. The skin effect problem for a round wire is physically the same as for a flat conductor. The mathematical development is different because of the difference in the configurations. The following analysis for a round wire has been made independent of the case for the flat conductor so that either case may be omitted if desired. If both cases are studied either one can be taken up first.

The magnetic field due to current in a round wire is indicated in Fig. 14–4. There will be flux both inside and outside the wire which follows circular paths concentric with the center of the conductor. The flux outside the wire is set up by all the current in the wire, and any change in the outer flux will induce a voltage within the wire which will have the same effect over the whole cross section. There is a different story inside the wire. The current in the conductor outside the radius x in Fig. 14–4 does not help set up the flux inside the radius x. The voltage due to a change in flux inside the radius x will appear only within this radius; there will be no induced voltage outside the circle of radius x due to changes in the flux inside this radius.

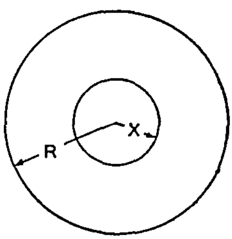

FIG. 14–4

The following mathematical treatment shows how the magnitude of the skin effect can be evaluated for round wires on the basis of the following restrictions:

1. Steady conditions have been established.
2. Currents and voltages are sine functions of time.
3. The return wire is far enough away so that its effect is negligible.
4. The permeability of the wire and of the space around the wire is constant and uniform.

Let D be the current density in the wire a distance x from the center. Then the total current within the radius x is

$$i_x = 2\pi \int_0^x xD\,dx. \tag{14.40}$$

This current is available to set up flux along the circle having radius x. The magnetic intensity is

$$H = \frac{i_x}{2\pi x} = \frac{1}{x}\int_0^x xD\,dx. \tag{14.41}$$

The flux density is

$$B = \mu H = \frac{\mu}{x}\int_0^x xD\,dx. \tag{14.42}$$

Figure 14–5 shows a longitudinal section of unit length of the wire. The flux density in the element $ABCD$ can be taken as the value obtained in equation (14.42). Therefore the total flux in this area will be

$$\Delta\phi = B(\text{area } ABCD) = B\Delta x$$

$$= \frac{\mu\Delta x}{x}\int_0^x xD\,dx. \tag{14.43}$$

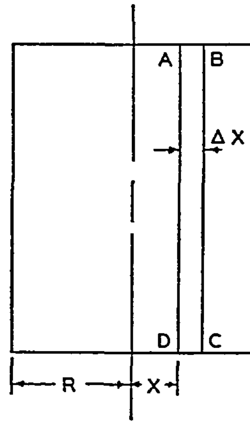

FIG. 14–5

The voltage induced in the path $ABCD$ will be

$$\frac{d(\Delta\phi)}{dt} = \frac{\mu\Delta x}{x}\int_0^x \frac{\partial D}{\partial t}x\,dx. \tag{14.44}$$

The partial derivative is necessary since D depends on both time and position and is a function of t and x.

The resistance drop along AD will be ρD where ρ is the resistivity; and the resistance drop along BC where the current density has a different value from that on AD will be

$$\rho D + \rho\frac{\partial D}{\partial x}\Delta x.$$

The resistance drops along AB and CD are zero because the current is in the direction of the conductor. The total resistance drop around $ABCD$ must be equal to the voltage induced, or

$$\rho D + \rho\frac{\partial D}{\partial x}\Delta x - \rho D = \frac{\mu\Delta x}{x}\int_0^x \frac{\partial D}{\partial t}x\,dx \tag{14.45}$$

$$\rho\frac{\partial D}{\partial x} = \frac{\mu}{x}\int_0^x \frac{\partial D}{\partial t}x\,dx. \tag{14.46}$$

Multiply both sides by x.

$$\rho x\frac{\partial D}{\partial x} = \mu\int_0^x \frac{\partial D}{\partial t}x\,dx. \tag{14.47}$$

Differentiate with respect to x to eliminate the integral.

$$\rho x\frac{\partial^2 D}{\partial x^2} + \rho\frac{\partial D}{\partial x} = \mu x\frac{\partial D}{\partial t}\cdot \tag{14.48}$$

This equation can be written for convenience

$$x^2 \frac{\partial^2 D}{\partial x^2} + x \frac{\partial D}{\partial x} - x^2 \frac{\mu}{\rho} \frac{\partial D}{\partial t} = 0. \tag{14.49}$$

If $(D_1 + iD)$ satisfies the above equation, where D_1 and D are both real and $i^2 = -1$, D_1 and D will each satisfy the equation. This can be shown as follows: Since $(D_1 + iD)$ satisfies the equation, we have

$$x^2 \frac{\partial^2 (D_1 + iD)}{\partial x^2} + x \frac{\partial (D_1 + iD)}{\partial x} - x^2 \frac{\mu}{\rho} \frac{\partial (D_1 + iD)}{\partial t} = 0. \tag{14.50}$$

This becomes

$$x^2 \frac{\partial^2 D_1}{\partial x^2} + ix^2 \frac{\partial^2 D}{\partial x^2} + x \frac{\partial D_1}{\partial x} + ix \frac{\partial D}{\partial x} - x^2 \frac{\mu}{\rho} \frac{\partial D_1}{\partial t} - ix^2 \frac{\mu}{\rho} \frac{\partial D}{\partial t} = 0. \tag{14.51}$$

$$\left[x^2 \frac{\partial^2 D_1}{\partial x^2} + x \frac{\partial D_1}{\partial x} - x^2 \frac{\mu}{\rho} \frac{\partial D_1}{\partial t}\right] + i\left[x^2 \frac{\partial^2 D}{\partial x^2} + x \frac{\partial D}{\partial x} - x^2 \frac{\mu}{\rho} \frac{\partial D}{\partial t}\right] = 0. \tag{14.52}$$

Therefore

$$x^2 \frac{\partial^2 D_1}{\partial x^2} + x \frac{\partial D_1}{\partial x} - x^2 \frac{\mu}{\rho} \frac{\partial D_1}{\partial t} = 0 \tag{14.53}$$

and

$$x^2 \frac{\partial^2 D}{\partial x^2} + x \frac{\partial D}{\partial x} - x^2 \frac{\mu}{\rho} \frac{\partial D}{\partial t} = 0 \tag{14.54}$$

showing that each D_1 and D satisfy the equation.

Since the currents and voltages are sine functions in time, we let

$$D_1 + iD = y\, e^{i\omega t}$$

and try to find a value for y in terms of x that will satisfy the above equation. Having found y in terms of x, the magnitude of the imaginary part of $y\, e^{i\omega t}$ will be the solution we seek and will be, if $y = y_1 + iy_2$,

$$D = y_1 \sin \omega t + y_2 \cos \omega t.$$

If we substitute $y\, e^{i\omega t}$ in equation (14.49), we get

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - x^2 \frac{i\omega\mu}{\rho} y = 0. \tag{14.55}$$

These are ordinary derivatives since y is a function of x alone. This

equation can be recognized as Bessel's equation if we replace $-i\omega\mu/\rho$ by a^2, obtaining

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + a^2x^2y = 0. \tag{14.56}$$

This has for the general solution $y = K_1J_0(ax) + K_2N_0(ax)$. Since $N_0(ax)$ becomes infinite as x approaches zero, the integration constant K_2 must be zero to keep the current density at the center of the wire finite, as we know it is. Therefore the solution is $y = KJ_0(ax)$, and the current density we seek is the imaginary part of $KJ_0(ax)\,e^{i\omega t}$, where K is a constant of integration which can be evaluated from the boundary conditions. The solution just obtained has the disadvantage that the constant a is not a real constant but has an imaginary part not equal to zero.

Let $\omega\mu/\rho = m^2$ in equation (14.55) and obtain

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - ix^2m^2y = 0. \tag{14.57}$$

This leads to the modified Bessel's functions. See section 10.14. The general solution is

$$y = K(\text{ber}\, mx + i\,\text{bei}\, mx). \tag{14.58}$$

Then D is the imaginary part of $K(\text{ber}\, mx + i\,\text{bei}\, mx)\,e^{i\omega t}$

$$D = K(\text{ber}\, mx \sin \omega t + \text{bei}\, mx \cos \omega t). \tag{14.59}$$

The total current is given by

$$i = 2\pi \int_0^R xD\,dx = 2\pi K \sin \omega t \int_0^R x\,\text{ber}\,(mx)\,dx + 2\pi K \cos \omega t \int_0^R x\,\text{bei}\,(mx)\,dx \tag{14.60}$$

$$= \frac{2\pi K \sin \omega t}{m^2} \int_0^R mx\,\text{ber}\,(mx)\,d(mx) + \frac{2\pi K \cos \omega t}{m^2} \int_0^R mx\,\text{bei}\,(mx)\,d(mx). \tag{14.61}$$

The two integrals in equation (14.61) are evaluated in equations (10.38) and (10.37), giving

$$i = \frac{2\pi K \sin \omega t}{m^2} mR\,\text{bei}'\,(mR) - \frac{2\pi K \cos \omega t}{m^2} mR\,\text{ber}'\,(mR) \tag{14.62}$$

$$= \frac{2\pi KR}{m}[\text{bei}'\,(mR) \sin \omega t - \text{ber}'\,(mR) \cos \omega t] \tag{14.63}$$

$$= \frac{2\pi KR}{m}\left[\text{bei}'\,(mR) \sin \omega t - \text{ber}'\,(mR) \sin\left(\omega t + \frac{\pi}{2}\right)\right. \tag{14.64}$$

In vector form we have

$$\mathbf{I} = \frac{2\pi KR}{m\sqrt{2}} [\text{bei}' (mR) - i \text{ ber}' (mR)]. \tag{14.65}$$

The voltage drop in the unit length of wire equals the resistance drop at the surface, or

$$v = \rho K [\text{ber} (mR) \sin \omega t + \text{bei} (mR) \cos \omega t] \tag{14.66}$$

$$= \rho K \left[\text{ber} (mR) \sin \omega t + \text{bei} (mR) \sin \left(\omega t + \frac{\pi}{2} \right) \right] \cdot \tag{14.67}$$

In vector form this is

$$\mathbf{V} = \frac{\rho K}{\sqrt{2}} [\text{ber} (mR) + i \text{ bei} (mR)]. \tag{14.68}$$

The impedance is found by dividing **V** by **I** and is

$$\mathbf{Z} = R_{ac} + iX = \frac{\rho m}{2\pi R} \frac{\text{ber} (mR) + i \text{ bei} (mR)}{\text{bei}' (mR) - i \text{ ber}' (mR)} \cdot \tag{14.69}$$

Multiply numerator and denominator by bei$'$ $(mR) + i$ ber$'$ (mR) and take the real part

$$R_{ac} = \frac{\rho m}{2\pi R} \frac{\text{ber} (mR) \text{ bei}' (mR) - \text{bei} (mR) \text{ ber}' (mR)}{\text{bei}'^2 (mR) + \text{ber}'^2 (mR)} \cdot \tag{14.70}$$

Now the resistance to direct current is

$$R_{dc} = \frac{\rho}{\pi R^2} \cdot$$

Therefore the ratio of R_{ac} to R_{dc} is

$$\frac{R_{ac}}{R_{dc}} = \frac{mR}{2} \frac{\text{ber} (mR) \text{ bei}' (mR) - \text{bei} (mR) \text{ ber}' (mR)}{\text{bei}'^2 (mR) + \text{ber}'^2 (mR)} \cdot \tag{14.71}$$

The following values can be used for copper wire:

$$\begin{aligned} \mu &= 1.257\ 10^{-8} \text{ henry}, \\ \omega &= 2\pi f, \\ \rho &= 1.7241\ 10^{-6} \text{ ohm-cm}, \\ mR &= 0.214R\sqrt{f}, \end{aligned}$$

where R is in cm and f is in cycles per second.

PROBLEMS ON CHAPTER 14

1. Show by direct substitution that D as defined in equation (14.29) is a solution of the differential equation (14.12). *Note:* m is defined in equation (14.20).

2. A flat conductor is made of copper 1 in. wide and 0.1 in. thick. If it is used in a 200,000-cycle circuit how does the a-c resistance compare with the d-c resistance?

3. If we arbitrarily say that the "skin" carries 85.9 per cent of the total current, how thick is the skin in problem 2?

4. How thick is the skin in problem 2 if the frequency is 500,000 cycles?

5. Solve problem 2 for an iron conductor. (Conductivity of iron is 17.4 per cent of that for copper; permeability of iron is 1500 times that of copper.)

6. Solve problem 3 for an iron conductor.

7. Solve problem 4 for an iron conductor.

8. If the skin of the conductor carries 85.9 per cent of the total current, what per cent is carried by the core in problem 2? Why is this more than 14.1 per cent?

9. Show that the d-c resistance of the skin of a flat conductor is equal to *half the a-c resistance.*

10. A copper wire has a diameter of 0.05 in. and a resistance to direct current of 10 ohms. Compute the resistance for several frequencies and plot a curve showing how the resistance varies with frequency. Be sure the computations give a point on the curve above 20 ohms. At what frequency will the a-c resistance of the wire be double the d-c resistance?

11. Repeat problem 10 for a wire 0.1 in. in diameter.

ANSWERS TO PROBLEMS

CHAPTER 1

3. $\dfrac{k^2(k^2-1)}{4!}\,d_0$

5. 0.56712, 0.56794, 0.57696

7. 0.56629, 0.57696, 0.58266, 0.58834

9. 0

11. $\dfrac{1}{\omega^2}\left(b_1 - c_2 + \dfrac{11}{12}\,d_2\right)$

13. $\dfrac{dz}{dx} = -y$

15. $$y = \frac{(x-x_2)(x-x_3)y_1}{(x_1-x_2)(x_1-x_3)} + \frac{(x-x_1)(x-x_3)y_2}{(x_2-x_1)(x_2-x_3)} + \frac{(x-x_1)(x-x_2)y_3}{(x_3-x_1)(x_3-x_2)}$$

17. $$\begin{aligned}\frac{dy}{dx_1} = {} & \frac{y_1}{x_1-x_2} + \frac{y_1}{x_1-x_3} + \frac{y_1}{x_1-x_4} + \frac{y_1}{x_1-x_5} \\ & + \frac{(x_1-x_3)(x_1-x_4)(x_1-x_5)y_2}{(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} \\ & + \frac{(x_1-x_2)(x_1-x_4)(x_1-x_5)y_3}{(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} \\ & + \frac{(x_1-x_2)(x_1-x_3)(x_1-x_5)y_4}{(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} \\ & + \frac{(x_1-x_2)(x_1-x_3)(x_1-x_4)y_5}{(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}\end{aligned}$$

19. 0.6094, 0.6435

CHAPTER 2

1.

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

3. 65, 62, 63, 64, 52, 53, 54, 74, 84
43, 63, 65
42, 43, 41, 52, 53, 51, 21, 31

5. 16523784, 124635

7.

52143	42315	2517436
12543	12345	2516437
12345		2513467
		1523467
		1253467
		1235467
		1234567

9. -1

11. 0

13. -60

15. -3

17. $\begin{matrix} 5 & 3 \\ 2 & 1 \end{matrix} \qquad \begin{matrix} 9 & 6 & 7 \\ -39 & 12 & 14 \\ -33 & -3 & 6 \end{matrix}$

27. 0

29. 0

31. $x = -13,\ y = 9$

33. $x = \dfrac{36}{107},\ y = -\dfrac{31}{107},\ z = \dfrac{21}{107}$

35. 18

37. 36; 225; 400; 225; 36; 1

39. 3

41. 3

43. not consistent

45. $x = \dfrac{z+1}{3},\ y = \dfrac{7z+4}{3}$

47. $x = \frac{5z}{3}, y = \frac{2z}{3}$

49. $x = -\frac{z + 5w}{10}, y = \frac{14z}{5}$; x and w cannot be expressed in terms of y and z.

CHAPTER 3

3. Period $= k\sqrt{LC}$ **11.** Energy $= kLI^2$ **13.** Velocity $= k\sqrt{gh}$

15. Energy $= \epsilon E^2 hf\left(\frac{A}{h^2}\right)$ Energy $= \frac{k\epsilon E^2 A}{h}$

17. $f\left(\frac{F}{\rho V^2 L^2}, \frac{S}{L^2}, \frac{\mu}{\rho LV}\right) = \text{constant}$

19. $f\left(\frac{F}{D^4 n^2 \rho}, \frac{S}{nD}, \frac{\mu}{\rho n D^2}, \frac{g}{Dn^2}\right) = \text{constant}$

21. $n' = n\sqrt{\frac{D}{D'}}, S' = S\sqrt{\frac{D'}{D}}, F' = F\left(\frac{D'}{D}\right)^3$

CHAPTER 4

1. (a) 3.61 cis 56°19′ (b) 3.61 cis 33°41′
(c) 3.61 cis −33°41′ (d) 3.61 cis 146°19′
(e) 3.61 cis 213°41′ (f) 9.43 cis 148°
(g) 10 cis 0° (h) 13.5 cis 221°59′
(i) 8.37 cis 49°36′ (j) 7.36 cis 41°58′
(k) 7.95 cis −10°40′ (l) 20 cis −90°
(m) 20.1 cis 185°10′ (n) 20.0 cis −92°52′
(p) 15.13 cis −82°24′ (q) 15.13 cis −97°36′

3. $6 + i8$
5. $1 + i0$
7. $-2 + i7$
9. $14.18 + i3.16$
11. $0.04 - i9.93$
13. $6.67 + i2.06$
15. $2.51 + i3.53$
17. $-6.93 + i9.00$
19. $2.66 - i5.78$
21. 15 cis 30°
23. 24 cis 180°
25. $18.85 + i0$
27. $-33.45 - i16.62$
29. 1.5 cis −5°
31. 2 cis 50°
33. cis 112°38′
35. 2.17 cis −71°52′
37. 13 cis 112°38′
39. 8 cis −90°
41. 1.899 cis 28°10′, 1.899 cis −151°50′
43. 1.866 cis 9°, 1.866 cis 81°, 1.866 cis 153°, 1.866 cis −135°, 1.866 cis −63°

Note: answers to the problems involving complex powers may differ from the listed answers by a factor $e^{2\pi}$.

45. 1.821 cis −100°24′
47. 314.8 cis 15°51′
49. 0.1038 cis 17°14′
51. 0.645 cis 64°51′
53. 5.28, 5.28 cis 72°, 5.28 cis 144°, 5.28 cis −144°, 5.28 cis −72°
55. 1, cis 72°, cis 144°, cis −144°, cis −72°
57. 1.189 cis −22°30′, 1.189 cis 157°30′
59. 0.265 cis 55°19′, 0.265 cis 127°19′, 0.265 cis −160°41′, 0.265 cis −88°41′, 0.265 cis −16°41′
67. $1.684 + i0.380$
69. $1.282 - i2.55$
71. $3.00 - i1.52$
73. $2.06 - i0.876$
75. cis −73°31′
77. 8103 cis 41°4′
79. −1
89. 0.47943
91. 1.0002
93. 1.01810
95. 0.22436
107. $-1.034 + i0.0430$
109. $-0.1923 - i0.302$
111. $0.842 + i0.1137$
113. 0.325 cis −70°22′

117. $0.296, 1.016, 1.317 + i1.571, 1.822 - i1.235, -2.371 - i1.184$
119. $0.310, 1.199 + i1.571, i1.107, 0.092 - i1.277, -0.067 + i1.743$

CHAPTER 5

1. 3
3. 137
13. $2y^3 - 9y^2 + 18y - 270 = 0$
15. $5y^5 - 56y^2 - 16y + 320 = 0$
17. $5y^4 - y^2 - 2y - 7 = 0$
19. $10y^3 - 2y^2 + 3y - 2 = 0$
21. $10y^5 - y^4 - 7y^3 + 5 = 0$
23. $5y^4 + 40y^3 + 114y^2 + 158y + 73 = 0$
25. $y^3 - 2y - 11 = 0, (y = x - 1)$
27. $2y^4 - 0.18775y^2 + 0.03128125y - 3.00143984375 = 0, (y = x + 0.125)$
29. 2.4
31. $\frac{5}{6}$
33. $\frac{10}{17}$
35. -2.4
39. $-\frac{7}{12}$
41. $11, \frac{10}{11}$; No negative roots.
43. $\frac{5}{2}, \frac{2}{3}; -\frac{3}{2}, -5$
45. $-2.49, 1.83, 0.657$
47. $2.61, -1.305 - i0.726, -1.305 + i0.726$
49. $2.053, 1.0264 + i2.484, 1.0264 - i2.484$
51. $2.536, -0.518 + i1.901, -0.518 - i1.901$

CHAPTER 6

1. $f'(c) \neq 0, f''(c)$ not restricted
3. $f'(c) \neq 0, f''(c)$ not restricted; or $f'(c) = 0, f''(c)$ not restricted
5. a is simple, b is simple, c is 4-fold or triple
7. Triple root -2
9. Double root 3, $i\sqrt{5}, -i\sqrt{5}$
11. No
13. $x = \pm\sqrt{\dfrac{-5 \pm \sqrt{37}}{2}}$
15. 1.244
17. 1.275
19. 1.1656
21. 4.424
23. -12
25. $-1.06 + i2.68, -1.06 - i2.68, 0.121$
27. $-1.598, 1.000, 1.799 + i0.716, 1.799 - i0.716$
29. $-8.556, 1.472, -0.453 + i0.951, -0.453 - i0.951$

CHAPTER 7

3. $b_1 = \dfrac{N}{\pi} \qquad b_2 = \dfrac{N}{2\pi}$

5. $y = \dfrac{3N}{4} - \dfrac{2N}{\pi^2}\sum_{1,3,5} \dfrac{1}{n^2}\cos nx - \dfrac{N}{\pi}\sum_{1,2,3}\dfrac{1}{n}\sin nx$

7. $y = \dfrac{N}{2} - \dfrac{4N}{\pi^2}\sum_{1,3,5}\dfrac{1}{n^2}\cos\dfrac{n\pi x}{2}$

9. $y = \dfrac{N}{2} - \dfrac{16N}{\pi^2}\sum_{2,6,10}\dfrac{1}{n^2}\cos\dfrac{2n\pi x}{5}$

11. $y = \dfrac{3N}{4} - \dfrac{N}{\pi^2}\sum_{1,3,5}\dfrac{\sqrt{4 + \pi^2 n^2}}{n^2}\cos(nx - \alpha_n) - \dfrac{N}{\pi}\sum_{2,4,6}\dfrac{1}{n}\cos\left(nx - \dfrac{\pi}{2}\right)$

$\tan\alpha_n = \dfrac{\pi n}{2}$

13. $y = \frac{3N}{4} - \frac{N}{\pi^2} \sum_{1,3,5} \frac{\sqrt{4 + \pi^2 n^2}}{n^2} \cos (n\pi x + \alpha_n) + \frac{N}{\pi} \sum_{2,4,6} \frac{1}{n} \cos \left(n\pi x - \frac{\pi}{2}\right)$

$\tan \alpha_n = \frac{n\pi}{2}$

15. $y = \frac{N}{2} - \frac{4N}{\pi^2} \sum_{1,3,5} \frac{1}{n^2} \sin \left(\frac{n\pi x}{2} + \frac{\pi}{2}\right)$

17. $y = \frac{3N}{4} - \frac{4N}{\pi^2} \sum_{1,3,5} \frac{1}{n^2} \sin \left(\frac{n\pi x}{2} + \frac{\pi}{2}\right) - \frac{8N}{\pi^2} \sum_{2,6,10} \frac{1}{n^2} \sin \left(\frac{n\pi x}{2} + \frac{\pi}{2}\right)$

19. Same as problem 9.

21. $y = \frac{8N}{\pi^2} \sum_{1,5,9} \frac{1}{n^2} \cos \left(\frac{2n\pi x}{7} - \frac{\pi}{2}\right) - \frac{8N}{\pi^2} \sum_{3,7,11} \frac{1}{n^2} \cos \left(\frac{2n\pi x}{7} - \frac{\pi}{2}\right)$

25. $y = - \sum_{2,4,6} \frac{2N}{n\pi} \sin \frac{nx}{2} + \sum_{1,5,9} \left(\frac{4N}{\pi^2 n^2} - \frac{2N}{n\pi}\right) \sin \frac{nx}{2}$

$\qquad + \sum_{3,7,11} \left(\frac{2N}{n\pi} - \frac{4N}{n^2\pi^2}\right) \sin \frac{nx}{2}$

27. $y = \sum_{1,5,9} \left(\frac{2N}{n\pi} + \frac{4N}{n^2\pi^2}\right) \sin \frac{n\pi x}{2} + \sum_{3,7,11} \left(\frac{2N}{n\pi} - \frac{4N}{n^2\pi^2}\right) \sin \frac{n\pi x}{2}$

$\qquad + \sum_{2,4,6} \frac{2N}{n\pi} \sin \frac{n\pi x}{2}$

29. $y = \frac{8N}{\pi^2} \sum_{1,5,9} \frac{1}{n^2} \sin \frac{n\pi x}{4} - \frac{8N}{\pi^2} \sum_{3,7,11} \frac{1}{n^2} \sin \frac{n\pi x}{4}$

31. $y = \frac{8N\sqrt{2}}{\pi^2} \sum_{1,3,9,11} \frac{1}{n^2} \sin \frac{n\pi x}{4} - \frac{8N\sqrt{2}}{\pi^2} \sum_{5,7,13,15} \frac{1}{n^2} \sin \frac{n\pi x}{4}$

33. $y = \frac{16N(\sqrt{2} - 1)}{\pi^2} \sum_{1,9} \frac{1}{n^2} \sin \frac{n\pi x}{5} + \frac{16N(\sqrt{2} + 1)}{\pi^2} \sum_{3,11} \frac{1}{n^2} \sin \frac{n\pi x}{5}$

$\qquad - \frac{16N(\sqrt{2} + 1)}{\pi^2} \sum_{5,13} \frac{1}{n^2} \sin \frac{n\pi x}{5} + \frac{16N(1 - \sqrt{2})}{\pi^2} \sum_{7,15} \frac{1}{n^2} \sin \frac{n\pi x}{5}$

35. $y = \frac{8N}{\pi^2} \sum_{1,5,9} \frac{1}{n^2} \sin \frac{2n\pi x}{7} - \frac{8N}{\pi^2} \sum_{3,7,11} \frac{1}{n^2} \sin \frac{2n\pi x}{7}$

37. 4.530, 1.901

39. 0.648, −0.031

CHAPTER 8

1. $y \frac{dy}{dx} + x = 0$

3. $(x^2 - 1) \left(\frac{dy}{dx}\right)^2 + x^2 = 0$

5. $3 \frac{dy}{dx} \left(\frac{d^2y}{dx^2}\right)^2 = \left[\left(\frac{dy}{dx}\right)^2 + 1\right] \frac{d^3y}{dx^3}$

7. $2x^3 + 9y^2 = C$

9. $y = Cx - 4x^2$

11. $\ln (xy) + 3y = C$

13. $(x^3 + 1)y - 6 \ln x = C$
15. $2xy - 1 = Cx^2$
17. $\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left[1 + \left(\frac{y}{x}\right)^2\right] = \ln x + C$
19. $y = C_1e^{-x} + C_2$
21. $\frac{2}{3}y^{\frac{3}{2}} = C_1x + C_2$
33. $y = C_1e^{-x} + C_2e^{-2x}$
35. $y = C_1 + C_2x + C_3e^{-2x} + C_4xe^{-2x}$
37. $y = C_1 \cos x + C_2 \sin x$
39. $y = \frac{1}{1-e}e^{-x} + \frac{e}{e-1}e^{-2x}$
41. $y = \cos x + \frac{2 - \cos 1}{\sin 1}\sin x$

CHAPTER 9

1. $y = C_1e^{-x} + C_2e^{-3x} + \frac{1}{8}e^x + \frac{1}{3}x - \frac{4}{9}$
3. $y = C_1 + C_2 \cos 3x + C_3 \sin 3x + \frac{1}{27}x^3 - \frac{2}{81}x$
5. $y = C_1e^x + C_2e^{-x} + C_3e^{-2x} + \frac{1}{12}e^{2x} + \frac{1}{6}xe^x - \frac{1}{20}x \sin 2x$
$\quad + \frac{1}{20}\cos 2x - \frac{1}{25}\sin 2x - \frac{13}{100}\cos 2x$
7. $y = C_1 + C_2e^{-x} + C_3xe^{-x} + C_4x^2e^{-x} - \frac{1}{18}x^3e^{-x} + \frac{1}{8}e^x$
$\quad + \frac{1}{3}x^3 - 3x^2 + 12x$
9. $y = C_1e^{-2x} + C_2e^x \sin 3x + C_3e^x \cos 3x + \frac{17}{580}\sin 2x - \frac{1}{580}\cos 2x$
11. $x = 2C_1e^{3t} + 2C_2e^{-3t} + C_3e^{2t} + C_4e^{-2t} - \frac{11}{36}t - \frac{2}{9}$
$y = -3C_1e^{3t} - 3C_2e^{-3t} + C_3e^{2t} + C_4e^{-2t} - \frac{5}{12}t - \frac{2}{3}$
13. $x = 2C_1e^{-3t}\sin 6t - 2C_2e^{-3t}\cos 6t + \frac{t}{15} + \frac{1}{75} + \frac{492}{1825}\sin 2t$
$\quad - \frac{144}{1825}\cos 2t$
$y = C_2e^{-3t}\sin 6t + C_1e^{-3t}\cos 6t - \frac{t}{15} + \frac{2}{225} + \frac{147}{1825}\sin 2t$
$\quad + \frac{46}{1825}\cos 2t$

CHAPTER 10

1. 3,628,800
3. 1,307,674,368,000
5. 1.3293
7. −3.5448
9. −0.94528
11. −0.06002

CHAPTER 11

1. 2i − 11j + 13k; −6i + 12j − 4k; 4i − j − 9k
3. i + j + k; i − j − k; −i + j − k; −i − j + k
5. 30° north of west
9. 48; i + 29j − 5k; 103; −i − 29j + 5k; 54; 2i + 58j − 10k
11. 0
13. 4.242i′ + 1.155j′ − 2.577k

CHAPTER 12

5. $(n - 1)r^{n-3}(x + y + z) + 3r^{n-1}$
$(1 - n)r^{n-3}\,[\mathbf{i}x(y + z) + \mathbf{j}y(z + x) + \mathbf{k}z(x + y)]$

CHAPTER 13

1. 448
3. 103
5. $$y = \frac{9M\sqrt{3}}{2\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi t}{L}\sqrt{\frac{T}{\rho}} + \frac{1}{4}\sin\frac{2\pi x}{L}\cos\frac{2\pi t}{L}\sqrt{\frac{T}{\rho}} - \frac{1}{16}\sin\frac{4\pi x}{L}\cos\frac{4\pi t}{L}\sqrt{\frac{T}{\rho}} - \frac{1}{25}\sin\frac{5\pi x}{L}\cos\frac{5\pi t}{L}\sqrt{\frac{T}{\rho}}\cdots\right]$$
7. $\frac{L}{3}$ from one end.
9. $$y = \frac{2vL}{\pi^2}\sqrt{\frac{\rho}{T}}\left[\sin\frac{\pi x}{L}\sin\frac{\pi t}{L}\sqrt{\frac{T}{\rho}} - \frac{2}{9}\sin\frac{3\pi x}{L}\sin\frac{3\pi t}{L}\sqrt{\frac{T}{\rho}} + \frac{1}{25}\sin\frac{5\pi x}{L}\sin\frac{5\pi t}{L}\sqrt{\frac{T}{\rho}} + \frac{1}{49}\sin\frac{7\pi x}{L}\sin\frac{7\pi t}{L}\sqrt{\frac{T}{\rho}}\cdots\right]$$
11. 32,800
13. Nickel 1.00
Brass 0.84
Steel 0.82
Bronze 0.77
15. 0.868
17. 1.037
19. 0.434
21. Second: 1.150; third: 1.200; fourth: 1.227
23. Second: 2.30; third: 2.40; fourth: 2.46

CHAPTER 14

3. 0.0116″
5. 139
7. 0.000455″
11. 32,250 cycles

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